# Solutions to Homework Assignment 25

**MATH 256-01**
**Section 5.4, Page 259**
**Problems: 1-18, 20**

1. Since $P(x)$ has only $x = 0$ as a zero, $x_0 = 0$ is the only singular point. Since $(x-0)\cdot\dfrac{1-x}{x} = 1-x$ is analytic and $(x-0)^2\cdot\dfrac{x}{x} = x^2$ is analytic, $x_0 = 0$ is a regular singular point.

2. The zeros of $P(x)$ are $x = 0$ and $x = 1$. $p(x) = \dfrac{2}{x(1-x)^2}$ and $q(x) = \dfrac{4}{x^2(1-x)^2}$. Now $xp(x) = \dfrac{2}{(1-x)^2}$ is analytic at $x_0 = 0$ and so is $x^2q(x) = \dfrac{4}{(1-x)^2}$. This means that $x_0 = 0$ is a regular singular point. On the other hand, $(x-1)p(x) = \dfrac{2}{x(1-x)}$ is not analytic at $x_0 = 1$, so $x_0 = 1$ is an irregular singular point.

3. The zeros of $P(x)$ are 0 and 1. $xp(x) = \dfrac{(x-2)}{x(1-x)}$ is not analytic at $x_0 = 0$, so $x_0 = 0$ is an irregular singular point. $(x-1)p(x) = \dfrac{(x-2)}{x^2}$ is analytic at $x_0 = 1$ and so is $(1-x)^2q(x) = -\dfrac{3(1-x)}{x}$, so $x = 1$ is a regular singular point.

4. The zeros of $P(x)$ are 0 and $\pm1$. $xp(x) = \dfrac{2}{x^2(1-x^2)}$ is not analytic at $x_0 = 0$, so $x_0 = 0$ is an irregular singular point. $(x-1)p(x) = -\dfrac{2}{x^3(1+x)}$ is analytic at $x_0 = 1$ and so is $(x-1)^2q(x) = -\dfrac{2(x-1)}{x^3(1+x)}$, so $x = 1$ is a regular singular point. Similarly, $x_0 = -1$ is a regular singular point.

5. The zeros of $P(x)$ are $\pm1$, both repeated. We have $p(x) = \dfrac{x}{(1-x)(1+x)^2}$ and $q(x) = \dfrac{1}{(1+x)(1-x)^2}$. Since $(x-1)p(x) = \dfrac{x}{(1+x)^2}$ and $(x-1)^2q(x) = \dfrac{1}{1+x}$ are both analytic at $x = 1$, $x = 1$ is a regular singular point. Since $(1+x)p(x) = \dfrac{x}{(1-x)(1+x)}$ is not analytic at $x = -1$, $x = -1$ is an irregular singular point.

6. We have $xp(x) = 1$ and $x^2q(x) = x^2 - v^2$, so $x = 0$ is a regular singular point.

7. $(x+3)p(x) = -2x$ and $(x+3)^2q(x) = (x+3)(1-x)^2$, so $x = -3$ is a regular singular point.

8. The singular points are $x = 0, \pm1$. We have $xp(x) = \dfrac{1}{1-x^2}$ and $x^2q(x)$ is a polynomial, so both are analytic at $x = 0$, making $x = 0$ a regular singular point. $(x-1)p(x) = -\dfrac{1}{x(1+x)}$, but $(x-1)^2q(x) = \dfrac{2}{x(1+x)^2(1-x)}$ is not analytic at $x = 1$, so $x = 1$ is an irregular singular point. $(x+1)p(x) = -\dfrac{1}{x(1-x)}$ and $(x+1)^2q(x) = \dfrac{2}{x(1-x)^3}$ are both analytic at $x = -1$, so $x = -1$ is a regular singular point.

9. The singular points are $x = -2, x = 1$. Since $(x+2)p(x) = \dfrac{3}{x+2}$ is not analytic at $x = -2$, $x_0 = -2$ is an irregular singular point. Since $(x-1)p(x) = \dfrac{3(x-1)}{(x+2)^2}$ and $(x-1)^2q(x) = -2\dfrac{x-1}{x+2}$ are both analytic at $x_0 = 1$, $x_0 = 1$ is a regular singular point.

10. The singular points are $x = 0$ and $x = 3$, and both are easily seen to be regular.

11. The singular points are $x = -2, 1$ (zeros of $x^2 + x - 2$), and it is easy to see that both are regular.

12. The only singular point is $x = 0$. Since $xp(x) = e^x$ and $x^2q(x) = 3x\cos x$ are both analytic at $x = 0$, $x_0 = 0$ is a regular singular point.

13. Since $p(x)$ is not defined at $x = 0$, $x = 0$ is a singular point. Since $x\ln|x|$ is not defined at $x = 0$ either, it is an irregular singular point.

14. The only singular point is $x_0 = 0$. We have $xp(x) = \dfrac{e^x - 1}{x} = \dfrac{\left(1 + x + \frac{1}{2}x^2 + \ldots\right) - 1}{x} = 1 + \dfrac{1}{2}x + \dfrac{1}{3!}x^2 + \ldots$ is analytic, and $x^2q(x) = e^{-x}\cos x$, so $x = 0$ is a regular singular point.

15. The only singular point is $x_0 - 0$. $xp(x) = \dfrac{-3\sin x}{x} = -3\cdot\dfrac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots}{x} = -3\left(1 - \dfrac{1}{3!}x^2 + \dfrac{1}{5!}x^4 - \ldots\right)$ is analytic and $x^2q(x) = 1 + x^2$ is analytic, so $x_0 = 0$ is a regular singular point.

16. $p(x) = \dfrac{1}{x}$ and $q(x) = \dfrac{\cot x}{x}$. $\cot x$ is a quotient of the two analytic functions $\cos x$ and $\sin x$, so it is analytic wherever $\sin x \neq 0$. Thus, the singular points are at $x = \pm n\pi$ and $x = 0$.

    We have $xp(x) = 1$. Write $q(x) = \dfrac{\cos x}{\sin x/x}$. The denominator is analytic, and we saw above that its series is $1 - \dfrac{1}{3!}x^2 + \dfrac{1}{5!}x^4 - \ldots$, which is nonzero at $x = 0$. Thus, $x^2q(x)$ is also analytic, so $x_0 = 0$ is a regular singular point.

    Since $\sin x$ is periodic with period $2\pi$ and $\sin(x + \pi)$ is just $-\sin x$, the behavior will be the same at every other singular point. Therefore, all of the singular points are regular.

17. The singular points are at $x = \pm n\pi$. We have $(x \mp n\pi)p(x) = \dfrac{x}{(-1)^n \sin(x \mp n\pi)/(x \mp n\pi)}$, so the denominator has the same behavior as in Number 16 (and is therefore analytic). Similarly, $(x \mp n\pi)^2q(x)$ is analytic, so each of $x \pm n\pi$ is a regular singular point.

18. This looks much like the last one except that $p(x)$ and $q(x)$ will have an extra factor of $x$ in the denominator. That means that the singular points other than zero will behave as before and be regular singular points, but $x = 0$ will be in irregular singular point since the denominator of $xp(x)$ is just $\sin x$.

20. It is easy to see that $x = 0$ is a regular singular point. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$. We get

$$
\begin{aligned}
&2x^2y'' + 3xy' - (1 + x)y \\
&= \sum_{n=2}^{\infty} 2a_n n(n-1)x^n + \sum_{n=1}^{\infty} 3a_n n x^n - \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1}x^n \\
&= (-a_0) + (3a_1 - a_1 - a_0)x + \sum_{n=2}^{\infty}(2a_n n(n-1) + 3a_n n - a_n - a_{n-1})x^n \\
&= 0.
\end{aligned}
$$

    This gives $a_0 = 0$ and $a_1 = 0$, so all of the other coefficients will also be zero since they are linear combinations of these two. Thus, there are no nonzero solutions of this form.