Solutions to Homework Assignment 26

MATH 256-01
Section 5.5, Page 265
Problems: 1-15 odd, 18, 19

1. Let $y = x^r$. We get $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$, and $\alpha = 4, \beta = 2$. Thus, the differential equation becomes $x^r(r^2 + (4-1)r + 2) = 0$. Therefore, $r^2 + 3r + 2 = 0$, so $r = -1$ or $r = -2$. The general solution is $y = c_1 x^{-1} + c_2 x^{-2}$. We do not need $|x|$ since c_1 can absorb the sign of x.

3. Let $y = x^r$. We have $\alpha = -3$ and $\beta = 4$, so proceeding as usual gives us $x^r(r^2 + (-3-1)r + 4) = 0$, and we get the repeated root $r = 2$. The general solution is thus $y(x) = c_1 x^2 + c_2 x^2 \ln|x|$. (The x^2 obviates the need for an absolute value.)

5. We have $\alpha = -1$ and $\beta = 1$. Thus $r^2 + (-1-1)r + 1 = 0$, so $r = 1, 1$. The general solution is $c_1 x + c_2 x \ln|x|$. The constants can absorb the sign of x.

7. We have $\alpha = 6$ and $\beta = -1$. Thus $r^2 + (6-1)r - 1 = 0$, so $r = \dfrac{-5 \pm \sqrt{29}}{2}$. The general solution is $c_1|x|^{(-5+\sqrt{29})/2} + c_2|x|^{(-5-\sqrt{29})/2}$.

9. We have $\alpha = -5$ and $\beta = 9$. Thus $r^2 + (-5-1)r + 9 = 0$, so $r = 3, 3$. The general solution is $c_1 x^3 + c_2 x^3 \ln|x|$.

11. We have $\alpha = 2$ and $\beta = 4$. Thus $r^2 + (2-1)r + 4 = 0$, so $r = \dfrac{-1 \pm \sqrt{-15}}{2}$. The general solution is $c_1|x|^{-1/2} \cos\left(\dfrac{\sqrt{15}}{2} \ln|x|\right) + c_2|x|^{-1/2} \sin\left(\dfrac{\sqrt{15}}{2} \ln|x|\right)$.

13. We have $\alpha = \dfrac{1}{2}$ and $\beta = -\dfrac{3}{2}$. Thus $r^2 + \left(\dfrac{1}{2} - 1\right) r - \dfrac{3}{2} = 0$, or $2r^2 - r - 3 = 0$, so $r = 3/2$ or $r = -1$. The general solution is $y(x) = c_1|x|^{3/2} + c_2 x^{-1}$. We require $y(1) = 1$, so $c_1 + c_2 = 1$. Also, since $1 > 0$, we may use $y(x) = c_1 x^{3/2} + c_2 x^{-1}$. This gives $y'(x) = \dfrac{3}{2} c_1 x^{1/2} - c_2 x^{-2}$, so $y'(1) = \dfrac{3}{2} c_1 - c_2 = 4$.

    Solving yields $c_1 = 2$ and $c_2 = -1$, so $y(x) = 2x^{3/2} - x^{-1}$.

15. This is the same as Number 3, so the general solution is $y(x) = c_1 x^2 + c_2 x^2 \ln|x|$. We have $y(-1) = c_1 = 2$. $y'(x) = 4x + c_2(2x \ln(-x) + x)$, so $y'(-1) = -4 + c_2(-1) = 3$, giving $c_2 = -7$. Thus, $y(x) = 2x^2 - 7x^2 \ln(-x)$.

18. We have $\alpha = 0$ and $\beta = \beta$. Thus $r^2 + (0-1)r + \beta = 0$, so $r = \dfrac{1 \pm \sqrt{1 - 4\beta}}{2}$.

    If $1 - 4\beta = 0$, then the solutions are $|x|^{1/2}$ and $|x|^{1/2} \ln|x|$; these both approach zero as $x \to 0$.

    Now suppose $1 - 4\beta > 0$. We need both solutions to be positive (so that x has a positive exponent). The negative root gives the smaller smaller solution, so it is sufficient to guarantee that this one is positive.

$$
\begin{aligned}
\frac{1 - \sqrt{1 - 4\beta}}{2} &> 0 \\
1 - \sqrt{1 - 4\beta} &> 0 \\
1 &> \sqrt{1 - 4\beta} \\
1 &> 1 - 4\beta \\
0 &> -4\beta \\
4\beta &> 0.
\end{aligned}
$$

Finally, if $1-4\beta < 0$, then we get $|x|^{1/2} \cos\left(\dfrac{\sqrt{4\beta - 1}}{2} \ln|x|\right)$ and a similar solution with a sine function. Both approach zero as $x \to zero$.

Thus, the only condition is that $\beta > 0$.

19. We may use our solution from Number 18 with $\beta = -2$: $y(x) = c_1 x^2 + c_2 x^{-1}$. Since $y(1) = 1$, $c_1 + c_2 = 1$. Since $y'(1) = \gamma$, $2c_1 - c_2 = \gamma$. Solving gives $c_1 = \dfrac{\gamma + 1}{3}$ and $c_2 = \dfrac{2 - \gamma}{3}$. In order to have a finite limit, we need to get rid of the x^{-1} term, so we choose $\gamma = 2$.