

Solutions to Homework Assignment 27

MATH 256-01
Section 5.6, Page 271
Problems: 1-11 odd

1. We first multiply through by $\frac{x}{2}$ to obtain the desired form: $x^2y'' + \frac{1}{2}xy' + \frac{1}{2}x^2y = 0$. It is easy to see that this has a regular singular point at $x = 0$. We have $p_0 = \frac{1}{2}$ and $q_0 = 0$ (with $p_1 = 0$ and $p_2 = 1/2$). The indicial equation is $r^2 + (1/2 - 1)r = 0$, or $2r^2 - r = 0$. Thus $r = 1/2$ or $r = 0$.

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. Then $y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$. We get

$$\begin{aligned} & 2x^2y'' + xy' + x^2y \\ &= \sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} \\ &= \sum_{n=0}^{\infty} (2a_n(n+r)(n+r-1) + a_n(n+r))x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r} \\ &= a_0(2r(r-1) + r)x^r + a_1(2(r+1)r + r + 1)x^{r+1} + \sum_{n=2}^{\infty} [a_n(n+r)(2(n+r-1) + 1) + a_{n-2}]x^{n+r} \\ &= 0. \end{aligned}$$

The first term is just the indicial equation, which we have already solved. The second implies that $a_1 = 0$ regardless of whether we choose $r = 1/2$ or $r = 0$. We have

$$a_n = -\frac{a_{n-2}}{(n+r)(2(n+r)-1)}.$$

With $r = 0$, we get $a_n = -\frac{a_{n-2}}{n(2n-1)}$. Thus $a_{2n+1} = 0$ for all n (since $a_1 = 0$).

Let $n = 2m$ now to consider only even n . We have $a_{2m} = -\frac{a_{2m-2}}{2m(4m-1)}$. Thus $a_2 = -\frac{a_0}{2 \cdot 3}$, $a_4 = -\frac{a_2}{2 \cdot 2 \cdot 7} = \frac{a_0}{2^2 \cdot 2 \cdot 3 \cdot 7}$, $a_6 = -\frac{a_4}{2 \cdot 3 \cdot 11} = -\frac{a_0}{2^3 \cdot 3! \cdot 3 \cdot 7 \cdot 11}$, \dots , $a_{2m} = \frac{(-1)^m a_0}{2^m m! \cdot 3 \cdot 7 \cdot \dots \cdot (4m-1)}$, \dots . The corresponding solution is

$$y_1(x) = 1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 - \dots + \frac{(-1)^m}{2^m m! \cdot 3 \cdot \dots \cdot (4m-1)}x^{2m} + \dots$$

With $r = 1/2$, we get $a_n = -\frac{a_{n-2}}{(n+1/2)(2(n+1/2)-1)} = -\frac{2a_{n-2}}{(2n+1)(2n)}$. Again, since $a_1 = 0$, all of the odd-numbered coefficients are zero, and we need only consider the even coefficients a_{2m} . This gives $a_{2m} = -\frac{a_{2m-2}}{2m(4m+1)}$.

$a_2 = -\frac{a_0}{2 \cdot 5}$, $a_4 = -\frac{a_2}{2 \cdot 2 \cdot 9} = \frac{a_0}{2^2 \cdot 2 \cdot 5 \cdot 9}$, \dots , $a_{2m} = \frac{(-1)^m a_0}{2^m m! \cdot 5 \cdot 9 \cdot \dots \cdot (4m+1)}$, \dots . The second solution is therefore

$$y_2(x) = x^{1/2} \left[1 - \frac{1}{10}x^2 + \frac{1}{360}x^4 - \dots + \frac{(-1)^m}{2^m m! \cdot 5 \cdot 9 \cdot \dots \cdot (4m+1)} + \dots \right].$$

3. Multiplying through by x gives $x^2y'' + xy = 0$, so $xp(x) = 0$ and $x^2q(x) = x$. Therefore, $p_0 = 0$ and $q_0 = 0$, so the indicial equation is just $r^2 + (0 - 1)r + 0 = 0$, or $r(r - 1) = 0$. The roots are 0 and 1. Since these differ by an integer, we will only find a solution for the larger one.

With $r = 1$, we have $y = \sum_{n=0}^{\infty} a_n x^{n+1}$, $y' = \sum_{n=0}^{\infty} a_n (n+1)x^n$, and $y'' = \sum_{n=1}^{\infty} a_n (n+1)nx^{n-1}$. The equation becomes

$$\begin{aligned} xy'' + y &= \sum_{n=1}^{\infty} a_n (n+1)nx^n + \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=1}^{\infty} a_n (n+1)nx^n + \sum_{n=1}^{\infty} a_{n-1}x^n \\ &= \sum_{n=1}^{\infty} [a_n(n+1)(n) + a_{n-1}]x^n \\ &= 0. \end{aligned}$$

We have $a_n = -\frac{a_{n-1}}{n(n+1)}$. Thus $a_0 = a_0$, $a_1 = -\frac{a_0}{2 \cdot 1}$, $a_2 = -\frac{a_1}{3 \cdot 2} = \frac{a_0}{3 \cdot 2 \cdot 2 \cdot 1}$, $a_3 = -\frac{a_2}{4 \cdot 3} = -\frac{a_0}{4!3!}$, \dots , $a_n = \frac{(-1)^n a_0}{(n+1)!n!}$, \dots . The solution is

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{3!2!}x^3 - \frac{1}{4!3!}x^4 + \dots + \frac{(-1)^n}{(n+1)!n!}x^{n+1} + \dots$$

5. Dividing through by 3 gives $x^2y'' + \frac{2}{3}xy' + \frac{1}{3}x^2y = 0$. We have $xp(x) = \frac{2}{3}$ (so $p_0 = 2/3$) and $x^2q(x) = \frac{1}{3}x^2$ (so $q_0 = 0$). The indicial equation is therefore $r^2 + (2/3 - 1)r + 0 = 0$, or $r^2 - 1r/3 = 0$. Thus $r = 0$ or $r = 1/3$. These do not differ by an integer, so we need to find both solutions.

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. Then $y' = \sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2}$. We get

$$\begin{aligned} 3x^2y'' + 2xy' + x^2y &= \sum_{n=0}^{\infty} 3a_n (n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} 2a_n (n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} \\ &= \sum_{n=0}^{\infty} a_n (n+r)(3(n+r-1) + 2)x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r} \\ &= a_0r(3r-1)x^r + a_1(r+1)(3r+2)x^{r+1} + \sum_{n=2}^{\infty} [a_n(n+r)(3(n+r-1) + 2) + a_{n-2}]x^{n+r} \\ &= 0. \end{aligned}$$

The first term is the indicial equation, and the second implies that $a_1 = 0$. Now $a_n = -\frac{a_{n-2}}{(n+r)(3(n+r)-1)}$, so all of the odd-numbered terms will be zero. Taking $n = 2m$ for the others, we see that

$$a_{2m} = -\frac{a_{2m-2}}{(2m+r)(3(2m+r)-1)}.$$

$$r = 0 \text{ gives } a_0 = a_0, a_2 = -\frac{a_0}{2 \cdot 5}, a_4 = -\frac{a_2}{4 \cdot 11} = \frac{a_0}{2^2 \cdot 2! \cdot 5 \cdot 11}, \dots, a_{2m} = \frac{(-1)^m a_0}{2^m m! \cdot 5 \cdot 11 \cdot \dots \cdot (6m-1)}.$$

The solution is $y_2(x) = 1 - \frac{1}{10}x^2 + \frac{1}{440}x^4 - \dots + \frac{(-1)^m}{2^m m! \cdot 5 \cdot 11 \cdot \dots \cdot (6m-1)}x^{2m} + \dots$

$r = 1/3$ gives $a_0 = a_0, a_{2m} = -\frac{a_{2m-2}}{2m(6m+1)}$, so we can see that the general coefficient will be

$a_{2m} = \frac{(-1)^m a_0}{2^m m! \cdot 7 \cdot 13 \cdots (6m+1)}$. This solution is

$$y_1(x) = x^{1/3} \left[1 - \frac{1}{2 \cdot 7} x^2 + \frac{1}{2^2 2! \cdot 7 \cdot 13} x^4 - \cdots + \frac{(-1)^m}{2^m m! \cdot 7 \cdot 13 \cdots (6m+1)} x^{2m} + \cdots \right].$$

7. Multiply through by x to get $x^2 y'' + (1-x)xy' - xy = 0$. We have $xp(x) = 1-x$ and $x^2 q(x) = -x$, so $p_0 = 1$ and $q_0 = 0$. The indicial equation is $r^2 + (1-1)r + 0 = 0$, so $r = 0, 0$ and we have a repeated root.

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. Then $y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$. We get (since $r = 0$)

$$\begin{aligned} & x^2 y'' + (x-x^2)xy' - xy \\ &= \sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=1}^{\infty} a_n n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=1}^{\infty} a_{n-1}(n-1)x^n - \sum_{n=1}^{\infty} a_{n-1} x^n \\ &= (a_1 - a_0)x + \sum_{n=2}^{\infty} [a_n n(n-1) + a_n n - a_{n-1}(n-1) - a_{n-1}]x^n \\ &= 0. \end{aligned}$$

Thus $a_1 = a_0$ and for $n \geq 2$, $a_n = \frac{a_{n-1}}{n}$. Therefore, $a_2 = \frac{a_1}{2} = \frac{a_0}{2}, a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \dots, a_n = \frac{a_0}{n!}$. The solution is $y_1(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x$.

9. We have $xp(x) = -x-3$ and $x^2 q(x) = x+3$, so $p_0 = -3$ and $q_0 = 3$. The indicial equation is $r^2 + (-3-1)r + 3 = 0$, so $r = 3$ or $r = 1$. Since these differ by an integer, we will just find the root corresponding to $r = 3$.

Let $y = \sum_{n=0}^{\infty} a_n x^{n+3}$. Then $y' = \sum_{n=0}^{\infty} a_n (n+3) x^{n+2}$ and $y'' = \sum_{n=0}^{\infty} a_n (n+3)(n+2) x^{n+1}$. The equation becomes

$$\begin{aligned} & x^2 y'' - x(x+3)y' + (x+3)y \\ &= \sum_{n=0}^{\infty} a_n (n+3)(n+2)x^{n+3} - \sum_{n=0}^{\infty} a_n (n+3)x^{n+4} - \sum_{n=0}^{\infty} 3a_n (n+3)x^{n+3} + \sum_{n=0}^{\infty} a_n x^{n+4} + \sum_{n=0}^{\infty} 3a_n x^{n+3} \\ &= \sum_{n=0}^{\infty} a_n (n+3)(n+2)x^{n+3} - \sum_{n=1}^{\infty} a_{n-1}(n+2)x^{n+3} - \sum_{n=0}^{\infty} 3a_n (n+3)x^{n+3} + \sum_{n=1}^{\infty} a_{n-1} x^{n+3} + \sum_{n=0}^{\infty} 3a_n x^{n+3} \\ &= (6a_0 - 9a_0 + 3a_0)x^3 + \sum_{n=1}^{\infty} [a_n((n+3)(n+2) - 3(n+3) + 3a_n) + a_{n-1}(-(n+2) + 1)]x^{n+3} \\ &= \sum_{n=1}^{\infty} [a_n n(n+2) - a_{n-1}(n+1)]x^{n+3} \\ &= 0. \end{aligned}$$

This gives $a_n = \frac{(n+1)a_{n-1}}{n(n+2)}$. Thus $a_0 = a_0, a_1 = \frac{2a_0}{1 \cdot 3}, a_2 = \frac{3a_1}{2 \cdot 4} = \frac{2a_0}{2 \cdot 4}, a_3 = \frac{4a_2}{3 \cdot 5} = \frac{2a_0}{2 \cdot 3 \cdot 5}, a_4 = \frac{5a_3}{4 \cdot 6} = \frac{2a_0}{2 \cdot 3 \cdot 4 \cdot 6}, a_5 = \frac{6a_4}{5 \cdot 7} = \frac{2a_0}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 7}, \dots, a_n = \frac{2a_0}{n! \cdot (n+2)}, \dots$

The solution is $x^3 + \frac{2}{3}x^4 + \frac{1}{4}x^5 + \frac{4}{15}x^6 + \dots + \frac{2}{n!(n+2)}x^{n+3} + \dots$

11. I will let $t = x - 1$. Then $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = y'$; similarly, $\frac{d^2y}{dt^2} = y''$. The equation becomes $t(t+2)y'' - 2(t+1)y' + \alpha(\alpha+1)y = 0$. Let $y = \sum_{n=0}^{\infty} a_n t^{n+r}$. Then $y' = \sum_{n=0}^{\infty} a_n(n+r)t^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)t^{n+r-2}$. We get

$$\begin{aligned}
& -t(t+2)y'' - 2(t+1)y' + \alpha(\alpha+1)y \\
&= -\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)t^{n+r} - 2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)t^{n+r-1} - \sum_{n=0}^{\infty} 2a_n(n+r)t^{n+r} \\
&\quad - \sum_{n=0}^{\infty} 2a_n(n+r)t^{n+r-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n t^{n+r} \\
&= -\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)t^{n+r-1} - 2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)t^{n+r-1} - \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)t^{n+r-1} \\
&\quad - \sum_{n=0}^{\infty} 2a_n(n+r)t^{n+r-1} + \sum_{n=1}^{\infty} \alpha(\alpha+1)a_{n-1}t^{n+r-1} \\
&= (-2a_0r(r-1) - 2a_0r)t^{r-1} + \sum_{n=1}^{\infty} [a_{n-1}(-(n+r-1)(n+r-2) - 2(n+r-1) + \alpha(\alpha+1)) \\
&\quad + a_n(-2(n+r)(n+r-1) - 2(n+r))]t^{n+r-1} \\
&= -2a_0(r^2)t^{r-1} + \sum_{n=1}^{\infty} [a_{n-1}(-(n+r-1)(n+r) + \alpha(\alpha+1)) - 2a_n(n+r)^2]t^{n+r-1} \\
&= 0.
\end{aligned}$$

The indicial equation is $r^2 = 0$, so the roots are both 0. Also, $a_n = -\frac{n(n-1) - \alpha(\alpha+1)}{2n^2}a_{n-1}$.

Thus $a_1 = \frac{\alpha(\alpha+1)}{2 \cdot 1^2}a_0$, $a_2 = -\frac{2 - \alpha(\alpha+1)}{2 \cdot 2^2}a_1 = -\frac{\alpha(\alpha+1)(2 - \alpha(\alpha+1))}{2^2(2!)^2}a_0, \dots$,

$a_n = (-1)^n \frac{\alpha(\alpha+1)(2 - \alpha(\alpha+1)) \cdots (n(n-1) - \alpha(\alpha+1))}{2^n(n!)^2}a_0, \dots$. The solution is

$$\begin{aligned}
y_1(t) &= 1 + \frac{\alpha(\alpha+1)}{2 \cdot 1^2}t - \frac{\alpha(\alpha+1)(2 - \alpha(\alpha+1))}{2^2(2!)^2}t^2 + \dots \\
&\quad + (-1)^{n+1} \frac{\alpha(\alpha+1)(2 - \alpha(\alpha+1)) \cdots (n(n-1) - \alpha(\alpha+1))}{2^n(n!)^2}t^n + \dots,
\end{aligned}$$

so

$$\begin{aligned}
y_1(x) &= 1 + \frac{\alpha(\alpha+1)}{2 \cdot 1^2}(x-1) - \frac{\alpha(\alpha+1)(2 - \alpha(\alpha+1))}{2^2(2!)^2}(x-1)^2 + \dots \\
&\quad + (-1)^{n+1} \frac{\alpha(\alpha+1)(2 - \alpha(\alpha+1)) \cdots (n(n-1) - \alpha(\alpha+1))}{2^n(n!)^2}(x-1)^n + \dots
\end{aligned}$$