Solutions to Homework Assignment 28

MATH 256-01
Section 5.7, Page 278
Problems: 1-15 odd

1. The only singular point is $x = 0$, and it is clearly regular. Multiply through by x to get the equation in the right form: $x^2y'' + x(2x)y' + (6xe^x)y = 0$. We see that $xp(x) = 2x$, so $p_0 = 0$, and $x^2q(x) = xe^x = x + x^2 + \frac{x^3}{2!} + \ldots$, so $q_0 = 0$ as well. The indicial equation is $r^2 + (0-1)r + 0 = 0$, or $r^2 - r = 0$. Therefore, the exponents at the singularity are $r_1 = 1$ and $r_2 = 0$.

3. Both $x = 0$ and $x = 1$ are singular points, and both are regular. For $x = 0$, multiply through by $\frac{x}{x-1}$ to get $x^2y'' + x \cdot \frac{6x^2}{x-1}y' + (3x)y = 0$. We have $xp(x) = \frac{6x^2}{x-1} = -6x^2(1 + x + x^2 + \ldots)$, giving $p_0 = 0$, and $x^2q(x) = 3x$, so $q(x) = 0$. We again get $r^2 + (0-1)r + 0 = 0$, so $r_1 = 1$ and $r_2 = 0$.

   For $x = 1$, multiply through by $\frac{x-1}{x}$ to get $(x-1)^2y'' + (x-1)(6x)y' + \frac{3(x-1)}{x}y = 0$. We have $(x-1)p(x) = 6x = 6 + 6(x-1)$, so $p_0 = 6$. Also, $x^2q(x) = \frac{3}{x} = \frac{3(x-1)}{1+(x-1)} = 3(x-1)(1 - (x-1) + (x-1)^2 - \ldots)$, so $q_0 = 0$. The indicial equation is $r^2 + (6-1)r = 0$, so $r_1 = 0$ and $r_2 = -5$.

5. The only singularity is $x = 0$. Since $x \cdot \frac{2\sin x}{x^2} = 2\left(1 - \frac{x^2}{3!} + \ldots\right)$ is analytic and $x^2 \cdot \frac{-2}{x^2} = -2$ is analytic, $x = 0$ is a regular singular point. We have $xp(x) = 3 - 3\frac{x^2}{3!} + \ldots$, so $p_0 = 0$, and $x^2q(x) = -2$, so $q_0 = -2$. Thus the indicial equation is $r^2 + (3-1)r - 2 = 0$, giving $r_1 = \frac{-2+\sqrt{12}}{2} = -1 + \sqrt{3}$ and $r_2 = -1 - \sqrt{3}$.

7. The only singularity is $x = 0$. We have $xp(x) = \frac{1}{2}\left(1 + 1 + \frac{x^3}{3!} + \ldots\right)$, so $p_0 = 1$. (Remember, the coefficient of y' is $x(xp(x))$.) Also, $x^2q(x) = 1$, so $q_0 = 1$. The indicial equation is $r^2 + (1-1)r + 1 = 0$, so $r = \pm i$.

9. $x = 0$ and $x = 1$ are both singular points. However, since $x \cdot \frac{-(1+x)}{x^2(1-x)}$ is not analytic at $x = 0$, this is an irregular singular point. It is not hard to check that $x = 1$ is a regular singular point. Multiply through by $\frac{(1-x)}{x^2}$ to obtain $(x-1)^2y'' + (x-1) \cdot \frac{1+x}{x^2}y' - \frac{2(1-x)}{x}y = 0$. We have $(x-1)p(x) = \frac{1+x}{x^2}$. $\lim_{x\to 1} \frac{1+x}{x^2} = 2$, so $p_0 = 2$. (This is like evaluating the series for $(x-1)p(x)$ at $x = 1$, giving the constant term.) Also, $x^2q(x) = -\frac{2(1-x)}{x}$, which is 0 at $x = 1$, so $q_0 = 0$. The indicial equation is $r^2 + (2-1)r + 0 = 0$, so $r_1 = 0$ and $r_2 = -1$.

11. The singular points are $x = \pm 2$, and both are regular. For $x = 2$, multiply through by $-\frac{x-2}{x+2}$ to get $(x-2)^2y'' - (x-2)\frac{2x}{x+2}y' - \frac{3(x-2)}{x+2}y = 0$. We have $(x-2)p(x) = \frac{-2x}{x+2}$, and this is -1 if $x = 2$, so $p_0 = -1$. Also, $(x-2)^2q(x) = -\frac{3(x-2)}{x+2}$, which is 0 at $x = 2$, so $q_0 = 0$. The indicial equation is $r^2 + (-1-1)r + 0 = 0$, so $r_1 = 2$ and $r_2 = 0$.

    For $x = -2$, multiply through by $\frac{x+2}{2-x}$ to get $(x+2)^2y'' + (x+2)\frac{2x}{2-x}y' + \frac{3(x+2)}{2-x}y = 0$. This time, $(x+2)p(x) = \frac{2x}{2-x}$, so $p_0 = -1$, and $(x+2)^2q(x) = \frac{3(x+2)}{2-x}$, and $q_0 = 0$. The indicial equation is $r^2 + (-1-1)r + 0 = 0$, so $r_1 = 2, r_2 = 0$.

13. Multiply through by x to get $x^2y'' + xy' - xy = 0$. We have $xp(x) = 1$ and $x^2q(x) = -x$, so $p_0 = 1$ and $q_0 = 0$. The indicial equation is $r^2 + (1-1)r + 0 = 0$, so $r = 0, 0$ and we have a repeated root. For the first solution, let $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$. (Although $r = 0$, we will want a_n in terms of r later in order to determine y_2.) Then $y'(x) = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$. Substituting gives us

$$
\begin{aligned}
& xy'' + y' - y \\
&= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \sum_{n=-1}^{\infty} a_{n+1}(n+r+1)(n+r)x^{n+r} + \sum_{n=-1}^{\infty} a_{n+1}(n+r+1)x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= (a_1 - a_0)x^{r-1} + \sum_{n=0}^{\infty} [a_{n+1}(n+r+1)^2 - a_n]x^{n+r-1} \\
&= 0.
\end{aligned}
$$

Thus $a_1 = a_0$ and $a_{n+1} = \dfrac{a_n}{(n+r+1)^2}$. With $r = 0$, we get $y_1(x) = 1 + x + \dfrac{1}{2^2}x^2 + \dfrac{1}{(3!)^2}x^3 + \dots$.

The solution y_2 has the form $y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} a_n'(0)x^n$, so we need to differentiate each a_n with respect to r.

$a_1 = \dfrac{a_0}{(r+1)^2}$, so $a_1' = -\dfrac{2a_0}{(r+1)^3}$. Evaluating at $r = 0$ gives $a_1'(0) = -2a_0$.

$a_2 = \dfrac{a_1}{(r+2)^2} = \dfrac{a_0}{(r+2)^2(r+1)^2}$. Thus $ln(a_2) = \ln(a_0) - 2\ln(r+2) - 2\ln(r+1)$, so $\dfrac{a_2'}{a_2} = -\dfrac{2}{r+2} - \dfrac{2}{r+1}$. Therefore, $a_2'(0) = a_2(0)(-1-2) = -\dfrac{3}{4}$.

(I hope you remember logarithmic differentiation!)

$a_3 = \dfrac{a_2}{(r+3)^2} = \dfrac{a_0}{(r+3)^2(r+2)^2(r+1)^2}$. Thus $\dfrac{a_3'}{a_3} = -2\left(\dfrac{1}{r+3} + \dfrac{1}{r+2} + \dfrac{1}{r+1}\right)$, and $a_3'(0) = -2a_3(0) \cdot \dfrac{11}{6} = -\dfrac{11}{108}$.

We are only asked for the first three nonzero terms, and we now have them, but I think we could also find the general term. We have

$$y_2(x) = y_1(x)\ln(x) - 2x - \frac{3}{4}x^2 - \frac{11}{108} \dots.$$

15. We know from Number 3 that $r_1 = 1$ and $r_2 = 0$. These differ by an integer $N = 1$, so we will need to do some work. Let us first find the solution corresponding to $r_1 = 1$.

Let $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$. Then $y'(x) = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$. We get

$$
\begin{aligned}
&x(x-1)y'' + 6x^2y' + 3y \\
&= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} 6a_n(n+r)x^{n+r+1} + 3\sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)x^{n+r-1} - \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} \\
&\quad + \sum_{n=2}^{\infty} 6a_{n-2}(n+r-2)x^{n+r-1} + \sum_{n=1}^{\infty} 3a_{n-1}x^{n+r-1} \\
&= a_0(r)(r-1)x^{r-1} + (a_0 r(r-1) - a_1 r(r+1) + 3a_0)x^r + \sum_{n=2}^{\infty}[a_{n-1}((n+r-1)(n+r-2)+3) \\
&\quad - a_n(n+r)(n+r-1) + 6a_{n-2}(n+r-2)]x^{n+r-1} \\
&= 0.
\end{aligned}
$$

We get $a_1 = \frac{r^2-r+3}{r(r+1)}a_0 = \frac{3}{2}$ for $r = 1$ and $a_n = \frac{[(n+r-1)(n+r-2)+3]a_{n-1} + 6(n+r-2)a_{n-2}}{(n+r)(n+r-1)}$.
Thus $a_2 = \frac{((r+1)\cdot r+3)a_1 + 6ra_0}{(r+2)(r+1)} = \frac{\frac{r^2-r+3}{r^2+r}(r^2+r+3)+6r}{(r+2)(r+1)}a_0 = \frac{3r^2+15r+9}{2(r+2)(r+1)}a_0 = \frac{9}{4}a_0$ (with $r = 1$) and
$a_3 = \frac{[(r+2)(r+1)+3]a_2 + 6(r+1)a_1}{(r+3)(r+2)} = \frac{\frac{9}{4}(r^2+3r+5)+9(r+1)}{(r+3)(r+2)}a_0 = \frac{9(r^2+7r+9)}{4(r+3)(r+2)}a_0 = \frac{51}{16}$. We have

$$y_1(x) = x + \frac{3}{2}x^2 + \frac{9}{4}x^3 + \frac{51}{16}x^4 + \ldots.$$

We finished this one in class.