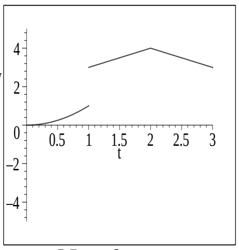
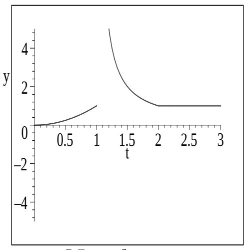
Solutions to Homework Assignment 30

MATH 256-01
Section 6.1, Page 298
Problems: 1-5, 7-10, 15-24

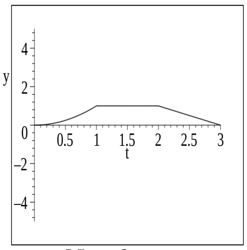
1. f is piecewise continuous on $[0, 3]$.
2. f is not continuous on $[0, 3]$.
3. f is continuous on $[0, 3]$.
4. f is piecewise continuous on $[0, 3]$.



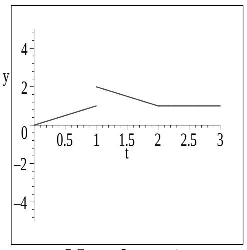
Number 1



Number 2



Number 3



Number 4

5. (a) $\mathcal{L}\{t\} = \int_0^\infty te^{-st}dt = \left(\frac{-1}{s}te^{-st} - \frac{1}{s^2}e^{-st}\right)\Big|_0^\infty = -\frac{1}{s^2}(0-1) = \frac{1}{s^2}$, $s > 0$.

   (b) $\mathcal{L}\{t^2\} = \int_0^\infty t^2e^{-st}dt$. Let $u = t^2, du = 2tdt, dv = e^{-st}dt, v = -\frac{1}{s}e^{-st}$. We get $-\frac{t^2}{s}e^{-st}\Big|_0^\infty + \frac{2}{s}\int_0^\infty te^{-st}dt = \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s^3}$.

   (c) With each additional factor of t, we will pick up a factor of s in the denominator and a factor of the exponent in the numerator, so $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$. (Use integration by parts once to compare $\mathcal{L}\{t^n\}$ with $\mathcal{L}\{t^{n-1}\}$.)

7. $\mathcal{L}\{\cosh bt\} = \mathcal{L}\left\{\frac{e^{bt}+e^{-bt}}{2}\right\} = \frac{1}{2}\left(\mathcal{L}\{e^{bt}\} + \mathcal{L}\{e^{-bt}\}\right) = \frac{1}{2}\left(\frac{1}{s-b} + \frac{1}{s+b}\right) = \frac{1}{2}\frac{2s}{s^2-b^2} = \frac{s}{s^2-b^2}$.

8. $\mathcal{L}\{\sinh bt\} = \mathcal{L}\left\{\frac{e^{bt}-e^{-bt}}{2}\right\} = \frac{1}{2}\left(\mathcal{L}\{e^{bt}\} - \mathcal{L}\{e^{-bt}\}\right) = \frac{1}{2}\left(\frac{1}{s-b} - \frac{1}{s+b}\right) = \frac{1}{2}\frac{2b}{s^2-b^2} = \frac{b}{s^2-b^2}$.

9. Let's establish a general formula for $G(s) = \mathcal{L}\{e^{at}f(t)\}$. Let $F(s) = \mathcal{L}\{f(t)\}$. We have $G(s) = \int_0^\infty e^{at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s-a)t}dt = F(s-a)$. **Record this!**

   Thus, $\mathcal{L}\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$ using the result and Number 7.

10. Using our theorem above, we compute $\mathcal{L}\{e^{at}\sinh t\} = \frac{b}{(s-a)^2-b^2}$.

15. This one doesn't require integration by parts; we may use our result above: $\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$.

16. $\mathcal{L}\{t\sin at\} = \int_0^\infty t\sin(at)e^{-st}dt$. Let $u = t\sin at, du = (\sin at + at\cos at)dt, dv = e^{-st}dt, v = -\frac{1}{s}e^{-st}$. We get $-\frac{1}{s}e^{-st}t\sin at\Big|_0^\infty + \frac{1}{s}\int_0^\infty(\sin at + at\cos at)e^{-st}dt = \frac{1}{s}\mathcal{L}\{\sin at\} + \frac{a}{s}\int_0^\infty t\cos(at)e^{-st}dt$. Now let $u = t\cos at, du = (\cos at - at\sin at)dt, dv = e^{-st}dt, v = -\frac{1}{s}e^{-st}$.

We have

$$\frac{1}{s}\mathcal{L}\{\sin at\} + \frac{a}{s}\left(-\frac{1}{s}t\cos(at)e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty (\cos at - at\sin at)e^{-st}dt\right)$$
$$= \frac{1}{s}\mathcal{L}\{\sin at\} + \frac{a}{s^2}\mathcal{L}\{\cos at\} - \frac{a^2}{s^2}\mathcal{L}\{t\sin at\}.$$

We may solve this for $\mathcal{L}\{t\sin at\}$, getting

$$\left(1+\frac{a^2}{s^2}\right)\mathcal{L}\{t\sin at\} = \frac{1}{s}\frac{a}{s^2+a^2} + \frac{a}{s^2}\frac{s}{s^2+a^2}$$
$$\frac{s^2+a^2}{s^2}\mathcal{L}\{t\sin at\} = \frac{2a}{s}\frac{1}{s^2+a^2}$$
$$\mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}.$$

That was a mess! I'm going to assign number 28 in section 6.2, which says that $\mathcal{L}\{(-t)^n f(t)\} = F^{(n)}(s)$. (Check that what we just did matches that!) For the rest of these, I'm just going to use that fact.

17. $\mathcal{L}\{t\cosh at\} = -\frac{d}{ds}\mathcal{L}\{\cosh at\} = -\frac{d}{ds}\frac{s}{s^2-a^s} = -\frac{1(s^2-a^2)-2s^2}{(s^2-a^2)^2} = \frac{s^2+a^2}{(s^2-a^2)^2}.$

18. $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$, where I've used the result for $\mathcal{L}\{f(t)e^{at}\}$.

19. $\mathcal{L}\{t^2\sin at\} = \frac{d^2}{ds^2}\left(\frac{s}{s^2-a^2}\right) = \frac{d}{ds}\left(\frac{2as}{(s^2+a^2)^2}\right) = \frac{2a(s^2+a^2)^2-4as^2(s^2+a^2)}{(s^2+a^2)^4} = \frac{2a((s^2+a^2)-2s^2)}{(s^2+a^2)^3} = \frac{2a(a^2-s^2)}{(s^2+a^2)^3}.$

20. $\mathcal{L}\{t^2\sinh at\} = \frac{d^2}{ds^2}\left(\frac{a}{s^2-a^2}\right) = \frac{d}{ds}\frac{-2as}{(s^2-a^2)^2} = \frac{2a(s^2-a^2)^2-4as^2(s^2-a^2)}{(s^2-a^2)^4} = \frac{2a((s^2-a^2)-2s^2)}{(s^2-a^2)^3} = -\frac{2a(a^2+s^2)}{(s^2+a^2)^3}.$

21. $\int_0^\infty \frac{1}{t^2+1}dt = \arctan t\Big|_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$

22. $\int_0^\infty te^{-t}$ converges since for sufficiently large t, $te^{-t} < e^{-t/2}$.

23. $\int_1^\infty \frac{e^t}{t^2}dt$ diverges since for sufficiently large t, $\frac{e^t}{t^2} > e^{t/2}$.

24. $\int_0^\infty e^{-t}\cos t dt$ converges since $|e^{-t}\cos t| \le e^{-t}$ for all t.