Solutions to Homework Assignment 31

MATH 256-01
Section 6.2, Page 307
Problems: 1-10, 11-23 odd, 25, 26

Note: "See number n" means find entry n in Table 6.2.1 on page 304.

1. The 3 is just a multiplicative factor, so we have $\frac{3}{2} \cdot \frac{2}{s^2+2^2}$, which is the transform of $\frac{3}{2}\sin(2t)$. (See Number 5.)

2. This is equal to $2 \cdot \frac{2!}{(s-1)^{2+1}}$, which is the transform of $2t^2e^t$. (See Number 11.)

3. Partial fraction decomposition gives $\frac{-2/5}{s+4} + \frac{2/5}{s-1}$, which is the transform of $-\frac{2}{5}e^{-4t} + \frac{2}{5}e^t$. (See Number 2.)

4. Partial fraction decomposition gives $\frac{9/5}{s-3} + \frac{6/5}{s+2}$, which is the transform of $\frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}$. (See Number 2.)

5. This is $\frac{2s+2}{(s+1)^2+2^2} = 2\frac{s+1}{(s+1)^2+2^2}$, which is the transform of $2e^{-t}\cos 2t$. (See Number 10.

6. This is $2\frac{s}{s^2-2^2} - \frac{3}{2} \cdot \frac{2}{s^2-2^2}$, which is the transform of $2\cosh 2t - \frac{3}{2}\sinh 2t$. (See Numbers 7 and 8.)

7. This is $2\frac{s-1}{(s-1)^2+1^2} + 3\frac{1}{(s-1)^2+1^2}$, which is the transform of $2e^t\cos t + 3e^t \sin t$. (See Numbers 9 and 10.) Here, I saw the denominator as $(s-1)^2+1^2$, so I wanted to locate either a constant in the numerator or a multiple of $s-1$ so that I could use 9 and/or 10.

8. We need a partial fraction decomposition: $\frac{8s^2-4s+12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$. Multiplying through by s and letting $s=0$ gives $A=3$. Multiplying through by s and letting $s \to \infty$ gives $8 = 3+B$, so $B=5$. Finally, multiplying through by $s-2i$ and letting $s=2i$ gives $\frac{-32-8i+12}{-8} = \frac{10i+C}{4i}$. This is $\frac{5}{2} + i = \frac{5}{2} - \frac{1}{4}Ci$. Therefore, $C=-4$, so we get $\frac{3}{s} + 5\frac{s}{s^2+2^2} - 2\frac{2}{s^2+2^2}$, which is the transform of $3 + 5\cos 2t - 2\sin 2t$. (See Numbers 1, 5, 6.)

9. This is $\frac{1-2s}{(s+2)^2+1^2} = 5\frac{1}{(s+2)^2+1^2} - 2\frac{s+2}{(s+2)^2+1^2}$, which is the transform of $5e^{-2t}\sin t - 2e^{-2t}\cos t$. (See Numbers 9 and 10.)

10. This is $2\frac{s+1}{(s+1)^2+3^2} - \frac{5}{3}\frac{3}{(s+1)^2+3^2}$, which is the transform of $2e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t$. (See Number 9 and 10.)

11. Applying the Laplace transform (and letting $Y(s) = \mathcal{L}\{y\}$) to both sides gives

$$\begin{aligned}
\mathcal{L}\{y'' - y' - 6y\} &= 0 \\
[s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 6Y(s) &= 0 \\
(s^2 - s - 6)Y(s) - s + 1 + 1 &= 0 \\
(s^2 - s - 6)Y(s) &= s - 2 \\
Y(s) &= \frac{s-2}{(s-3)(s+2)} \\
&= \frac{1/5}{s-3} + \frac{4/5}{s+2}.
\end{aligned}$$

Thus, using Number 2, we see that the solution is $\frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$.

13. We have

$$\begin{aligned}
\mathcal{L}\left\{y'' - 2y' + 2y\right\} &= 0 \\
[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 2Y(s) &= 0 \\
(s^2 - 2s + 2)Y(s) - 0 - 1 + 0 &= 0 \\
Y(s) &= \frac{1}{s^2 - 2s + 2} \\
&= \frac{1}{(s-1)^2 + 1^2}.
\end{aligned}$$

This is the transform of $e^t \sin t$ (using Number 9.)

15. We have

$$\begin{aligned}
\mathcal{L}\left\{y'' - 2y' - 2y\right\} &= 0 \\
[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] - 2Y(s) &= 0 \\
(s^2 - 2s - 2)Y(s) - 2s - 0 + 4 &= 0 \\
Y(s) &= \frac{2s - 4}{s^2 - 2s - 2} \\
&= \frac{2s - 4}{(s-1)^2 - (\sqrt{3})^2} \\
&= 2\frac{s - 1}{(s-1)^2 - (\sqrt{3})^2} - \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{(s-1)^2 - (\sqrt{3})^2}.
\end{aligned}$$

This is the transform of $2e^t \cosh \sqrt{3}t - \frac{2}{\sqrt{3}}e^t \sinh \sqrt{3}t$. Use Numbers 7, 8, and 14.

17. We have

$$\begin{aligned}
\mathcal{L}\left\{y^{(4)} - 4y''' + 6y'' - 4y' + y\right\} &= 0 \\
[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)]& \\
-4[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)]& \\
+6[s^2Y(s) - sy(0) - y'(0)]& \\
-4[sY(s) - y(0)] + Y(s) &= 0 \\
(s^4 - 4s^3 + 6s^2 - 4s + 1)Y(s) - s^2 - 1 + 4s - 6 &= 0 \\
Y(s) &= \frac{s^2 - 4s + 7}{(s-1)^4} \\
&= \frac{(s-1)^2 - 2s + 6}{(s-1)^4} \\
&= \frac{1}{(s-1)^2} + \frac{-2(s-1) + 4}{(s-1)^4} \\
&= \frac{1}{(s-1)^2} - 2\frac{1}{(s-1)^3} + 4\frac{1}{(s-1)^4} \\
&= \frac{1!}{(s-1)^2} - \frac{2!}{(s-1)^3} + \frac{2}{3}\frac{3!}{(s-1)^4}.
\end{aligned}$$

This is the transform of $te^t - t^2e^t + \frac{2}{3}t^3e^t$. Use Number 11.

19. We have

$$\begin{aligned}
\mathcal{L}\left\{y^{(4)} - 4y\right\} &= 0 \\
[s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)] - 4Y(s) &= 0 \\
(s^4 - 4)Y(s) - s^3 + 2s &= 0 \\
Y(s) &= \frac{s^3 - 2s}{(s^2-2)(s^2+2)} \\
&= \frac{s}{s^2+2}.
\end{aligned}$$

This is the transform of $\cos\sqrt{2}t$. (Use Number 6.)

21. We have

$$\begin{aligned}
\mathcal{L}\left\{y'' - 2y' + 2y\right\} &= \mathcal{L}\left\{\cos t\right\} \\
[s^2 Y(s) - s y(0) - y'(0)] - 2[sY(s) - y(0)] + 2Y(s) &= \frac{s}{s^2+1} \\
(s^2 - 2s + 2)Y(s) - s + 2 &= \frac{s}{s^2+1} \\
(s^2 - 2s + 2)Y(s) &= \frac{s}{s^2+1} + s - 2 \\
Y(s) &= \frac{s}{(s^2+1)(s^2-2s+2)} + \frac{s-2}{s^2-2s+2} \\
&= \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-2s+2} + \frac{s-2}{s^2-2s+2}.
\end{aligned}$$

Ignore the last two terms temporarily while we do a partial fraction decomposition. Multiplying through by s^2+1 and letting $s = i$ gives $\frac{i}{1-2i} = Ai + B$, so $Ai + B = \frac{i-2}{5}$. Therefore, $A = \frac{1}{5}$ and $B = -\frac{2}{5}$ (since we require $A, B, C,$ and D to be real). Multiplying through by s and letting $s \to \infty$ gives $0 = A + C$, so $C = -\frac{1}{5}$. Finally, putting $s = 1$ on both sides gives $\frac{1}{2} = \frac{1}{10} - \frac{1}{5} - \frac{1}{5} + D$, so $D = \frac{4}{5}$. The right-hand side is now

$$\begin{aligned}
&\frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} + \frac{1}{5}\frac{-s+4}{s^2-2s+2}\frac{s-2}{s^2-2s+2} \\
&= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} + \frac{1}{5}\frac{4s-6}{(s-1)^2+1^2} \\
&= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} + \frac{4}{5}\frac{s-1}{(s-1)^2+1^2} - \frac{2}{5}\frac{1}{(s-1)^2+1^2}.
\end{aligned}$$

This is the transform of $\frac{1}{5}\cos t - \frac{2}{5}\sin t + \frac{4}{5}e^t\cos t - \frac{2}{5}e^t\sin t$.

23. We have

$$
\begin{aligned}
\mathcal{L}\left\{y''+2y'+y\right\} &= \mathcal{L}\left\{4e^{-t}\right\} \\
[s^2Y(s)-sy(0)-y'(0)]+2[sY(s)-y(0)]+Y(s) &= \frac{4}{s+1} \\
(s^2+2s+1)Y(s)-2s+1-4 &= \frac{4}{s+1} \\
Y(s) &= \frac{4}{(s+1)^3}+\frac{2s+3}{(s+1)^2} \\
&= 2\frac{2!}{(s+1)^3}+2\frac{1}{s+1}+\frac{1!}{(s+1)^2}.
\end{aligned}
$$

This is the transform of $2t^2e^{-t}+2e^{-t}+te^{-t}$.

25. The Laplace transform of the right-hand side is $\int_0^1 te^{-st}dt = -\frac{1}{s}\left(te^{-st}+\frac{1}{s}e^{-st}\right)\Big|_0^1 = -\frac{1}{s}\left(e^{-s}+\frac{1}{s}e^{-s}-\frac{1}{s}\right) = \frac{1}{s^2}(1-(s+1)e^{-s})$. The right-hand side is

$$
\begin{aligned}
\mathcal{L}\left\{y''+y\right\} &= [s^2Y(s)-sy(0)-y'(0)]+Y(s) \\
&= (s^2+1)Y(s).
\end{aligned}
$$

Thus, $Y(s) = \frac{1-(s+1)e^{-s}}{s^2(s^2+1)}$.

26. The Laplace transform of the right-hand side is

$$
\begin{aligned}
\int_0^1 te^{-st}dt+\int_1^\infty e^{-st}dt &= \frac{1}{s^2}(1-(s+1)e^{-s})-\frac{1}{s}e^{-st}|_1^\infty \\
&= \frac{1}{s^2}(1-(s+1)e^{-s})-\frac{1}{s}(0-e^{-s}) \\
&= \frac{1}{s^2}(1-e^{-s}).
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\mathcal{L}\left\{y''+4y\right\} &= \frac{1}{s^2}(1-e^{-s}) \\
[s^2Y(s)-sy(0)-y'(0)]+4Y(s) &= \frac{1}{s^2}(1-e^{-s}) \\
(s^2+4)Y(s) &= \frac{1}{s^2}(1-e^{-s}) \\
Y(s) &= \frac{1-e^{-s}}{s^2(s^2+4)}.
\end{aligned}
$$