Solutions to Homework Assignment 34

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1. The general solution is $y = c_1 \cos x + c_2 \sin x$. Since $y(0) = 0$, we require $c_1 = 0$. Since $y'(\pi) = 1$, we require $c_2(-1) = 1$, so $c_2 = -1$. We therefore have the unique solution $y(x) = -\sin x$.

2. The general solution is $y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x$. $y' = -\sqrt{2}c_1 \sin \sqrt{2}x + \sqrt{2}c_2 \cos \sqrt{2}x$. Since $y'(0) = 1$, we require $\sqrt{2}c_2 = 1$; thus, $c_2 = \frac{1}{\sqrt{2}}$. Since $y'(\pi) = 0$, we require $-\sqrt{2}c_1 \sin \sqrt{2}\pi + \cos \sqrt{2}\pi = 0$; therefore, $c_1 = \frac{1}{\sqrt{2}} \cos \sqrt{2}\pi$. The general solution is $y(x) = \frac{1}{\sqrt{2}} \cot \sqrt{2}\pi \cos \sqrt{2}x + \frac{1}{\sqrt{2}} \sin \sqrt{2}x$.

3. The general solution is $y(x) = c_1 \cos x + c_2 \sin x$. Since $y(0) = 0$, we have $c_1 = 0$. Since $y(L) = 0$, we have $c_2 \sin L = 0$. This has no nontrivial solution if $L \neq n\pi$ for some integer n, and it has the nontrivial solution $c_2 \sin x$ if $L = n\pi$ for some integer π.

4. This has the same general solution as Number 3. Since $y'(0) = 1$, $c_2 = 1$. Thus $y(x) = c_1 \cos x + \sin x$. $y(L) = c_1 \cos L + \sin L = 0$, so $c_1 = -\tan L$ unless $\cos L = 0$. If $\cos L = 0$, then $\sin L \neq 0$, so there is no solution. In the first case, the solution is $y(x) = -\tan L \cos x + \sin x$.

5. We have $(D^2 + 1)D^2 y = 0$, which has general solution $c_1 + c_2 x + c_3 \cos x + c_4 \sin x$. The sine and cosine are solutions of the corresponding homogeneous equation, so we just have $y_p(x) = c_1 + c_2 x$. This gives $y_p'' = 0$, so $0 + c_1 + c_2 x = x$. Therefore, $c_1 = 0$ and $c_2 = x$.

   We have $y(x) = x + c_3 \cos x + c_4 \sin x$. Since $y(0) = 0$, this means $c_3 = 0$. The condition $y(\pi) = 0$ then implies that $\pi = 0$, so there is no solution. (But we got a nice review!)

6. We have $(D^2 + 2)D^2 y = 0$, so the solution has the form $c_1 + c_2 x + c_3 \cos x + c_4 \sin x$. The solutions of the corresponding homogeneous equation are $\sin \sqrt{2}x$ and $\cos \sqrt{2}x$, so $y_p = c_1 + c_2 x$ and $y_p'' = 0$. We get $2(c_1 + c_2 x) = x$, so $c_1 = 0$ and $c_2 = 1/2$. Now $y(x) = \frac{1}{2}x + c_3 \cos \sqrt{2}x + c_4 \sin \sqrt{2}x$.

   $y(0) = 0$, so $c_3 = 0$. $y(\pi) = 0 = \frac{\pi}{2} + c_4 \sin(\sqrt{2}\pi)$. Thus $c_4 = -\frac{\pi}{2 \sin \sqrt{2}\pi}$. Therefore, $y(x) = \frac{1}{2}x - \frac{\pi}{2 \sin \sqrt{2}\pi} \sin(\sqrt{2}x)$.

7. We have $(D^2 + 4)(D^2 + 1)y = 0$, giving a solution of the form $c_1 \cos 2x + c_2 \sin 2x + A \cos x + B \sin x$. The $\cos 2x$ and $\sin 2x$ terms solve the corresponding homogeneous equation, so $y_p = A \cos x + B \sin x$ and $y_p'' = -A \cos x - B \sin x$. Therefore,

   $\cos x = y_p'' + 4y_p = 3A \cos x + 3B \sin x$. This gives $A = \frac{1}{3}$ and $B = 0$, so $y(x) = \frac{1}{3} \cos x + c_1 \cos 2x + c_2 \sin 2x$.

   Now $y(0) = 0$, so $\frac{1}{3} + c_1 = 0$, and $c_1 = -\frac{1}{3}$. $y(\pi) = 0$, as well, so $-\frac{1}{3} + c_1 = 0$, and $c_1 = \frac{1}{3}$. Uh-oh! This has no solution.

8. We may reuse most of our work from the last problem; the only difference is that $A = 0$ and $B = \frac{1}{3}$. Thus $y(x) = \frac{1}{3} \sin x + c_1 \cos 2x + c_2 \sin 2x$. Now $y(0) = c_1 = 0$, so $c_1 = 0$. Also, $y(\pi) = 0$, which adds no new conditions. Thus $y(x) = \frac{1}{3} \sin x + c_2 \sin 2x$. Note that there are infinitely many solutions.

9. We may reuse our work from Number 7:$y(x) = \frac{1}{3} \cos x + c_1 \cos 2x + c_2 \sin 2x$.

   Since $y'(0) = 0$, we have $c_2 = 0$. Since $y'(\pi) = 0$, we have again that $c_2 = 0$. Therefore, the solution is $y(x) = \frac{1}{3} \cos x + c_1 \cos 2x$, and again there are infinitely many solutions.

10. We may again borrow from earlier work; we just replace the 2's in the prior problems with $\sqrt{3}$'s and the $\frac{1}{3}$ with a $\frac{1}{2}$: $y(x) = \frac{1}{2}\cos x + c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x$.

    Now $y'(0) = 0$ implies that $c_2 = 0$. $y'(\pi) = 0$ implies that $-c_1\sqrt{3}\sin\sqrt{3}\pi = 0$, so $c_1 = 0$ as well. Thus $y(x) = \frac{1}{2}\cos x$.

11. Assume first that $\lambda > 0$. The general solution is $y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. With $y(0) = 0$, we have $c_1 = 0$. With $y'(\pi) = 0$, we have $\sqrt{\lambda}c_2\cos(\sqrt{\lambda}\pi) = 0$. In order to get a nontrivial solution, we need to have $\sqrt{\lambda}\pi = \frac{(2n+1)\pi}{2}$ for some integer n. Thus, the eigenvalues are $\frac{(2n+1)^2}{4}$ for n a nonnegative integer. The corresponding eigenfunctions are $y_n(x) = c_2 \sin\frac{2n+1}{2}x$.

    If $\lambda = 0$, we have $y'' = 0$, so $y = c_1 + c_2x$. This cannot satisfy the boundary conditions, so $\lambda = 0$ is not an eigenvalue.

    If $\lambda < 0$, we have $y(x) = c_1e^{\sqrt{\lambda}x} + c_2e^{-\sqrt{\lambda}x}$. It is not hard to see that the only solution of this form is the trivial solution.

12. For $\lambda > 0$, the general solution is $y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. With $y'(0) = 0$, we have $c_2 = 0$. With $y(\pi) = 0$, we have $c_1 \cos\sqrt{\lambda}\pi = 0$. To get a nontrivial solution, we must have $\sqrt{\lambda}\pi = \frac{(2n+1)\pi}{2}$ for some integer n. Thus, the eigenvalues are $\frac{(2n+1)^2}{4}$ for n a nonnegative integer. The corresponding eigenfunctions are $y_n(x) = c_1 \cos\frac{2n+1}{2}x$.

    Again, there are no negative eigenvalues and zero is not an eigenvalue.

13. For $\lambda > 0$, the general solution is $y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. With $y'(0) = 0$, we have $c_2 = 0$. With $y'(\pi) = 0$, we have $-c_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi = 0$. To get a nontrivial solution, we must have $\sqrt{\lambda}\pi = n\pi$ for some integer n. Thus, the eigenvalues are n^2 for n a positive integer. The corresponding eigenfunctions are $y_n(x) = c_1 \cos nx$.

    If $\lambda = 0$, then $y(x) = c_1 + c_2x$, so $y'(0) = 0$ implies that $c_2 = 0$. Now $y(x) = c_1$ will also satisfy $y'(\pi) = 0$, so $y(x) = c_1$ is an eigenfunction for the eigenvalue 0.