

Solutions to Homework Assignment 10

MATH 345-01

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1,2,3,4

1. (a) $f(z) = \bar{z} = x - iy$. $u_x = 1 \neq -1 = v_y$.
 - (b) $f(z) = z - \bar{z} = 2x$. $u_x = 2 \neq 0 = v_y$.
 - (c) $u_x = 2, v_y = 2xy$. These are equal if $xy = 1$. However, $u_y = 0$ and $-v_x = -y^2$, and these are only equal if $y = 0$. Since $y = 0$ and $xy = 1$ are incompatible, f is not differentiable anywhere.
 - (d) $f(z) = e^x \cos y - ie^x \sin y$. $u_x = e^x \cos y, v_y = -e^x \cos y$. Thus $\cos y = 0$, so $y = \pi/2 + k\pi$ for some $k \in \mathbb{Z}$. On the other hand, $u_y = -e^x \sin y$ and $-v_x = e^x \sin y$. Thus $\sin y = 0$, so $y = m\pi$ for some $m \in \mathbb{Z}$. We can't have both!
2. (a) $u_x = 0 = v_y, u_y = -1 = -v_x$ and everything here is continuous. Therefore f' exists. Since $f'(z) = i$, the 'new' $u_x = 0 = v_y$ and $u_y = 0 = -v_x$. Again, these are continuous, so $f''(z)$ exists. In fact, $f''(z) = 0$.
 - (b) $f(z) = e^{-x} \cos y - e^{-x} \sin y$. Thus $u_x = -e^{-x} \cos y = v_y$ and $u_y = -e^{-x} \sin y = -v_x$. Since these are continuous everywhere, $f'(z)$ exists, and $f'(z) = -e^{-x} \cos y + ie^{-x} \sin y$. Repeat.
 - (c) (c) and (d) are similar to (a) and (b). Recall that $(\cosh x)' = \sinh x$ and $(\sinh x)' = \cosh x$.
3. (a) $f(z) = \frac{1}{z} = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$. Thus $u_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} = v_y$ and $u_y = -\frac{2xy}{(x^2 + y^2)^2} = -v_x$. Thus $f'(z)$ exists for $x^2 + y^2 \neq 0$, or $z \neq 0$, and its value is

$$\begin{aligned}
 f'(z) &= u_x + iv_x \\
 &= \frac{y^2 - x^2}{(x^2 + y^2)^2} + i \frac{2xy}{(x^2 + y^2)^2} \\
 &= -\frac{x^2 - y^2 - 2ixy}{(x^2 + y^2)^2} \\
 &= -\frac{(x - iy)^2}{(x^2 + y^2)^2} \\
 &= -\frac{1}{(x + iy)^2} \\
 &= -\frac{1}{z^2}.
 \end{aligned}$$

- (b) $u_x = 2x, v_y = 2y$. These are only equal for $y = x$. $u_y = 0 = -v_x$. Thus f is differentiable if $y = x$, and the derivative is $f'(z) = 2x$.
 - (c) $f(x) = xy + iy^2$. $u_x = y, v_y = 2y$; these are only equal for $y = 0$. $u_y = x, -v_x = 0$; these are only equal for $x = 0$. Thus $f'(z)$ exists only at $z = 0$ and $f'(z) = 0$ there.
4. (a) $f(z) = \frac{1}{r^4} e^{-4i\theta} = \frac{1}{r^4} \cos 4\theta - i \frac{1}{r^4} \sin 4\theta$. $ru_r = -\frac{4}{r^4} \cos 4\theta = v_\theta$ and $u_\theta = -\frac{4}{r^4} \sin 4\theta = -rv_r$, so f is differentiable. $f'(z) = e^{-i\theta} \left(-\frac{4}{r^5} \cos 4\theta + i \frac{4}{r^5} \sin 4\theta \right) = -\frac{1}{r^5} e^{-i\theta - 4i\theta} = -\frac{1}{z^5}$.
 - (b) The other parts are similar.