Solutions to Homework Assignment 23

MATH 345-01

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1,3,4,5,7

1. (a) $dz = 2ie^{i\theta}d\theta$. We get $\displaystyle\int_0^{\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}} 2ie^{i\theta}d\theta = 2e^{i\theta} + 2i\theta\Big|_0^{\pi} = -4 + 2i\pi$.
   (b) (b) is similar.
   (c) (c) is the sum of the answers in (a) and (b).
3. We did this in class.
4. $C : z(t) = t + it^3$ on $[-1, 1]$, so $dz = (1 + 3it^2)dt$. Notice also that $y < 0$ when $t < 0$ and $y > 0$ when $t > 0$. We get $\displaystyle\int_{-1}^{0} 1(1 + 3it^2)dt + \int_0^1 4t^3(1 + 3it^2)dt = (t + it^3)|_{-1}^{0} + (t^4 + 2it^6)|_0^1 = 2 + 3i$.
5. Let C be a smooth contour given by $z = z(t) on [a, b]$. Then $dz = z'(t)dt$, so $\int_C 1dz = \displaystyle\int_a^b u'(t)+iv'(t)dt = u(b) + iv(b) - u(a) - iv(a) = z(b) - z(a) = z_2 - z - 1$.
7. The given function is analytic on the complex plane minus the branch cut on the positive x-axis. This is a little tricky. If we change our branch cut to be at $-\pi/2$ and take $g(z) = z^{-1+i}$ on this branch, then $f(z) = g(z)$ when $0 < \arg z \leq \pi$. Thus, since they differ at only one point, their integrals will match. Likewise, if we take $h(z) = z^{-1+i}$ on the branch with cut at $\pi/2$, we have $h(z) = f(z)$ when $\pi \leq \arg z \leq 2\pi$ (except at 2π). Again their integrals will match over this region. Therefore $\int_C f(z)dz = \int_{C_1} g(z)dz + \int_{C_2} h(z)dz$. Indeed, if we keep in mind that at the initial point the argument of z is 0 and at the ending point it is 2π, we may simply compute these together:

$$\begin{aligned}
\int_C z^{-1+i}dz &= \int_C \exp[(-1+i)\log z]dz \\
&= \int_0^{2\pi} \exp[(-1+i)\log e^{i\theta}]ie^{i\theta}d\theta \\
&= i\int_0^{2\pi} e^{-i\theta-\theta}e^{i\theta}d\theta \\
&= i\int_0^{2\pi} e^{-\theta}d\theta \\
&= -i(e^{-2\pi}-1) \\
&= i(1-e^{-2\pi}).
\end{aligned}$$