Solutions to Homework Assignment 31

MATH 345

Section 68, Page 205

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1. We know $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$. Thus, for $z \neq 0$, we get $z^2 \sin(1/z^2) = \sum_{n=0}^{\infty} \frac{(-1)^n z^2 (1/z^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (1/z^2)^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}$.

5. Since neither term has a singularity in $|z| < 1$, we can just use the sum of their two McClaurin series (which are geometric): $f(z) = \frac{1}{z-1} - \frac{1}{z-2} = -\frac{1}{1-z} + \frac{1}{2-z} = -\frac{1}{1-z} + \frac{1/2}{1-(z/2)} = -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}(1/2)^n z^n = \sum_{n=0}^{\infty}(1/2^{n+1} - 1)z^n$.

   For D_2, since $\frac{1}{z-2}$ has no singularities in $|z| < 2$, we can reuse its McClaurin series. However, $\frac{1}{z-1}$ does have a singularity in $|z| < 2$ (namely, $z = 1$), so we find $a_n = \frac{1}{2\pi i}\int_C \frac{1/(z-1)}{z^{n+1}} dz$, where C is a simple closed curve in D_2 with $z = 1$ inside of it. Rather than trying to compute the integral, we will take advantage of the fact that $|1/z| < 1$ in D_2, as in Example 2 of Section 68, to rewrite $\frac{1}{z-1}$ as $\frac{1/z}{1-(1/z)} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=1}^{\infty}\frac{1}{z^n}$. Since both series converge in D_2, we get $f(z) = \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty}\frac{1}{z^n}$.

   Finally, on D_3 we may reason similarly except that this time both terms have singularities when $|z| > 2$. We get

$$\begin{aligned}
\frac{1}{z-1} - \frac{1}{z-2} &= \frac{1/z}{1-(1/z)} - \frac{1/z}{1-(2/z)} \\
&= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{z}\sum_{n=0}^{\infty}(2/z)^n \\
&= \sum_{n=1}^{\infty}\frac{1}{z^n} - \sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n} \\
&= \sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n}.
\end{aligned}$$