Solutions to Homework Assignment 34

MATH 345

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1. (a) $\dfrac{1}{z+z^2} = \dfrac{1}{z}\dfrac{1}{1-(-z)} = \dfrac{1}{z}(1 - z + z^2 - z^3 + \ldots) = \dfrac{1}{z} - 1 + z - z^2 + \ldots$, so $\operatorname*{Res}_{z=0} f(z) = 1$ (the coefficient of $1/z$).

   (c) $\dfrac{z - \sin(z)}{z} = \dfrac{z - \left(z - \frac{1}{6}z^3 + \ldots\right)}{z} = \dfrac{1}{6}z^2 + \ldots$. Since there is no $\dfrac{1}{z}$ term, $\operatorname*{Res}_{z=0} = 0$.

   (d) The Maclaurin series for $\tan z$ is $z + \dfrac{z^3}{3} + \dfrac{2z^5}{15} + \ldots$. For the reciprocal of this, perform long division to find $\cot(z) = \dfrac{1}{z} - \dfrac{z}{3} - \dfrac{z^3}{45} - \ldots$. Since we are also dividing by z^4, the residue will come from the z^3 term and is therefore $-\dfrac{1}{45}$.

   (e) The series for $\sinh z$ is $\displaystyle\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$, and the series for $\dfrac{1}{1-z^2}$ is $\displaystyle\sum_{n=0}^{\infty} z^{2n}$. Since we are also dividing by z^4, we need the 3rd-degree term of the product of those series: $\left(\displaystyle\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\right)\left(\displaystyle\sum_{n=0}^{\infty} z^{2n}\right) = \left(z + \dfrac{z^3}{6} + \dfrac{z^5}{120} + \cdots\right)(1 + z^2 + z^4 + \ldots) = z + z^3\left(1 + \dfrac{1}{6}\right) + \ldots$. Dividing by z^4 gives us $\dfrac{1}{z^3} + \dfrac{7}{6}\cdot\dfrac{1}{z} + \ldots$, so $\operatorname*{Res}_{z=0} f(z) = \dfrac{7}{6}$.

2. (a) $\dfrac{e^{-z}}{z^2} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{n-2}}{n!}$, so $\operatorname*{Res}_{z=0} f(z) = -1$. Thus $\int_C f(z)dz = 2\pi i(-1) = -2\pi i$.

   (b) $\dfrac{e^{-z}}{(z-1)^2} = \displaystyle\sum_{n=0}^{\infty} \frac{(-e^{-1})^n (z-1)^{n-2}}{n!}$, so $\operatorname*{Res}_{z=1} f(z) = -\dfrac{1}{e}$. Thus $\int_C f(z)dz = 2\pi i(-1/e) = -\dfrac{2\pi i}{e}$.

   (d) $\dfrac{z+1}{z^2-2z} = \dfrac{z+1}{z}\cdot\dfrac{1}{z-2}$, so there are two isolated singularities to consider: at $z=0$ and $z=2$. At $z=0$, we have

$$\begin{aligned}
\frac{1}{z}\left(\frac{z-2+3}{z-2}\right) &= \frac{1}{z}\left(1 + \frac{3}{z-2}\right) \\
&= \frac{1}{z}\left(1 - \frac{3/2}{1-(z/2)}\right) \\
&= \frac{1}{z}\left(1 - \frac{3}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \ldots\right)\right),
\end{aligned}$$

   so $\operatorname*{Res}_{z=0} f(z) = -\dfrac{1}{2}$.

At $z = 2$, we have

$$\begin{aligned}\frac{1}{z-2}\left(1+\frac{1}{z}\right) &= \frac{1}{z-2}\left(1+\frac{1}{2-(2-z)}\right)\\ &= \frac{1}{z-2}\left(1+\frac{1/2}{1-(2-z)/2}\right)\\ &= \frac{1}{z-2}\left(1+\frac{1}{2}\left(1-\frac{z-2}{2}+\frac{(z-2)^2}{2}+\ldots\right)\right),\end{aligned}$$

so $\operatorname*{Res}_{z=2} f(z) = \frac{3}{2}$. Therefore $\int_C f(z)dz = 2\pi i\left(-\frac{1}{2}+\frac{3}{2}\right) = 2\pi i$.

1. (a) Note that the singularities all satisfy $z^3 = 1$, so they all have modulus 1 and are therefore all inside C. Now $\frac{1}{z^2}f(1/z) = \frac{1}{z^2}\frac{1/z^5}{1-(1/z^3)} = \frac{1}{z^7-z^4} = -\frac{1}{z^4}\frac{1}{1-z^3} = -\frac{1}{z^4}(1+z^3+z^6+\ldots)$, so $\operatorname*{Res}_{z=0}\frac{1}{z^2}f(1/z) = -1$. Therefore, $\int_C f(z)dz = -2\pi i$.

   (c) We already know this is $2\pi i$ by our standard calculation (and because the residue at 0 is 1), but also $\frac{1}{z^2}\frac{1}{(1/z)} = \frac{1}{z}$, so $\operatorname*{Res}_{z=\infty}\frac{1}{z} = \operatorname*{Res}_{z=0}\frac{1}{z} = 1$, and we again get $\int_C f(z)dz = 2\pi i$.