

Solutions to Homework Assignment 35

MATH 345

Section 79, Page 242

1abc, 2ac

1. (a) $f(z) = ze^{1/z} = z \left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots \right)$, so the principal part is $\frac{1}{2z} + \frac{1}{6z^2} + \dots$. Thus $z = 0$ is an essential singularity.
 - (b) $f(z) = \frac{z^2}{1+z} = \frac{z^2 - 1 + 1}{1+z} = z - 1 + \frac{1}{1+z} = \frac{1}{1+z} - 2 + (1+z)$, so $z = -1$ is a simple pole.
 - (c) $f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{4!} - \dots$, so $z = 0$ is a removable singularity.
 - (e) $f(z) = \frac{1}{(2-z)^3} = -\frac{1}{(z-2)^3}$ is already a Laurent series; $z = 2$ is a pole of order 3.
2. (a) $f(z) = \frac{1 - \cosh z}{z^3} = \frac{1}{z^3} \left(1 - \left(1 + \frac{z^2}{2} + \frac{z^4}{4!} + \dots \right) \right) = -\frac{1}{2z} - \frac{z}{4!} - \dots$, so $z = 0$ is a simple pole and $\operatorname{Res}_{z=0} f(z) = -\frac{1}{2}$.
 - (c) $f(z) = \frac{e^{2z}}{(z-1)^2}$: Taking $g(z) = e^{2z}$, we find $g^{(n)}(1) = 2^n e^2$, so
$$f(z) = \frac{1}{(z-1)^2} \left(1 + 2e^2(z-1) + \frac{4e^2(z-1)^2}{2} + \frac{8e^2(z-1)^3}{3!} + \frac{16e^2(z-1)^4}{4!} + \dots \right) = \frac{1}{(z-1)^2} + \frac{2e^2}{z-1} + 2e^2 + \frac{4e^2(z-1)}{3} + \dots$$
 Thus f has a pole of order 2 at $z = 1$, and $\operatorname{Res}_{z=1} f(z) = 2e^2$.