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Theorem

The Cauchy-Goursat Theorem: *If f is analytic everywhere on and inside a simple closed contour C, then* $\int_C f(z)dz = 0.$

Big Picture: We will subdivide the region R enclosed by C into little squares where we can estimate $f(z)$ and integrate around those instead of around C. We will show that that substitution (the squares for C) does not change the result, and the estimate we get is "less than ϵ"-ish.
For this, we need to do a lot of work!

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2. Lemma 1: If $[a_n, b_n]$ is a sequence of intervals such that $[a_{n+1}, b_{n+1}]$ is either the left or right half $[a_n, b_n]$ for each $n = 0, 1, 2, \ldots$, then $\bigcap_n [a_n, b_n]$ is a single point.

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3. Lemma 2: If $\sigma_0, \sigma_1, \ldots$ is a sequence of squares with $\sigma_k = [a_k, b_k] \times [c_k, d_k]$ and $[a_k, b_k]$ and $[c_k, d_k]$ as in Lemma 1, then $\bigcap_n \sigma_n$ is a single point.

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4. Lemma 3: Let C be a positively oriented simple closed contour, and let R be C along with its interior. Assume that f is analytic on R. For every $\epsilon > 0$, R can be covered with a finite number of (partial) grid squares $S_1, S_2, \ldots, S_n$ such that each S_k contains a point z_k with the property that for all other $z \in S_k$,

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$$\delta_k = \begin{cases} \dfrac{f(z) - f(z_k)}{z - z_k} - f'(z_k) & z \neq z_k \\ 0 & z = z_k \end{cases}$$

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