Review of Integration

Definitions

1. **Curves:** Let $C : z(t) = x(t) + iy(t)$ define a curve in $\mathbb{C}$ for $t \in [a, b]$. We say C is
   - (a) an **arc** if x and y are continuous on $[a, b]$.
   - (b) **simple** if $z(s) \neq z(t)$ for $s \neq t$ in $[a, b]$.
   - (c) a **closed** curve if $z(a) = z(b)$.
   - (d) a **simple closed curve** or **Jordan arc** if it is simple except for $z(a) = z(b)$.
   - (e) **positively oriented** if it is a **simple closed curve** traversed in the counter-clockwise direction.
   - (f) a **differentiable arc** if x' and y' exist and are continuous on $[a, b]$.
   - (g) a **smooth** arc if it is a differentiable arc with $z'(t) \neq 0$ on (a, b).
   - (h) a **contour** if it is a piecewise smooth arc (finitely many pieces).
   - (i) a **simple closed contour** if it is a contour that is also a simple closed curve.

2. **Sets:** Let $D \subseteq \mathbb{C}$. We say D is
   - (a) **connected** if there is a polygonal path in D between any two points of D.
   - (b) a **domain** if it is a connected open set.
   - (c) a **region** if it is a domain along with some or possibly none of its boundary.
   - (d) **simply connected** if every simple closed contour $C \subseteq D$ satisfies $\text{int}(C) \subseteq D$.
   - (e) **multiply connected** if it is not simply connected.

Theorems

1. **(Antiderivative)** Let f be continuous in a domain D. The following are equivalent:
   - (a) f has an antiderivative F throughout D.
   - (b) Integrals over contours in D between two fixed points of D are independent of path ($\displaystyle\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$).
   - (c) $\int_C f(z)dz = 0$ for all closed contours $C \subseteq D$.

2. **(Cauchy-Goursat)** If f is analytic on and interior to a simple closed contour C, then $\int_C f(z)dz = 0$.

3. **(Cauchy-Goursat Corollary)** If f is analytic throughout a simply connected domain D, then for every closed contour $C \subseteq D$, $\int_C f(z)dz = 0$.

4. **(Deformation Principle)** Let C_1, C_2 be positively oriented simple closed contours with $C_1 \subseteq \text{int } C_2$. If f is analytic on and between C_1 and C_2, then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

5. **(Cauchy Integral Formula)** Let C be a positively oriented simple closed contour, and assume f is analytic on and inside C. If $z_0 \in \text{int } C$, then $f(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z - z_0}dz$.

6. **(Cauchy Integral Formula for Derivatives)** With the same hypotheses as in the CIF, $f^{(n)}(z_0) = \dfrac{n!}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z - z_0)^{n+1}}dz$.

7. **(Analyticity)** If f is analytic at a point, so are its derivatives of all orders.

8. **(Morera's)** Let f be continuous on a domain D. If $\int_C f(z)dz = 0$ for every closed contour $C \subseteq D$, then f is analytic on D.

9. **(Liouville's)** Every bounded entire function is constant.

10. **(Fundamental Theorem of Algebra)** Every polynomial in $\mathbb{C}[z]$ has a root in $\mathbb{C}$.

11. **(Maximum Modulus Principle)** If f is analytic and non-constant on a domain D, then $|f(z)|$ has no maximum in D.

12. **(MMP Corollary)** Let R be a closed and bounded region and that f is a function that is continuous on R and analytic and non-constant on the interior of R. Then the maximum value of $|f(z)|$ occurs on the boundary of R and not on the interior of R.