MATH 356-01
Solutions to Homework Assignment 1

1.3 (a) We have $m = 6$ and $n = 20$, so the corresponding root is

$$x = \sqrt[3]{\frac{20}{2} + \sqrt{\frac{20^2}{4} + \frac{6^3}{27}}} - \sqrt[3]{-\frac{20}{2} + \sqrt{\frac{20^2}{4} + \frac{6^3}{27}}} = \sqrt[3]{10 + 6\sqrt{3}} - \sqrt[3]{-10 + 6\sqrt{3}}.$$

(b) This time, $m = -15$ and $n = 4$. We get

$$x = \sqrt[3]{\frac{4}{2} + \sqrt{\frac{4^2}{4} + \frac{(-15)^3}{27}}} - \sqrt[3]{-\frac{4}{2} + \sqrt{\frac{4^2}{4} + \frac{(-15)^3}{27}}} = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{2 + \sqrt{-121}}.$$

This is $\sqrt[3]{2 - 11i} - \sqrt[3]{-2 + 11i}$. The hint suggests we consider $(2 + i)^3 = 2 + 11i$ and $(-2 + i)^3 = -2 + 11i$. Ta-da! We now have $\sqrt[3]{2 - 11i} - \sqrt[3]{-2 + 11i} = (2 + i) - (-2 + i) = 4$.

1.4 $i^{17} = i, i^{35} = -i$, and $i^{428347} = i^3 = -i$ (since $i^4 = 1$).

1.5 $z = 2 + 3i, w = 4 - 2i : z + w = (2 + 3i) + (4 - 2i) = (2 + 4) + (3 - 2)i = 6 + i$, $z - w = (2 + 3i) - (4 - 2i) = -2 + 5i, zw = (2 + 3i)(4 - 2i) = (2)(4) + (2)(-2i) + (3i)(4) + (3i)(-2i) = 14 + 8i, \frac{2 + 3i}{4 - 2i} = \frac{2 + 3i}{4 - 2i} \cdot \frac{4 + 2i}{4 + 2i} = \frac{2 + 16i}{20} = \frac{1}{10} + \frac{4}{5}i, \overline{z} = 2 - 3i, |z| = \sqrt{20}$.

1.6 $f = 2 + 3x, g = 4 - 2x : f + g = (2 + 3x) + (4 - 2x) = (2 + 4) + (3 - 2)x = 6 + x, f - g = (2 + 3x) - (4 - 2x) = -2 + 5x, fg = (2 + 3x)(4 - 2x) = (2)(4) + (2)(-2x) + (3x)(4) + (3x)(-2x) = 8 + 8x - 6x^2, \deg((f + g)^3) = \deg(6 + x)^3 = 3, \deg(f^3 + g^3) = \deg((2 + 3x)^3 + (4 - 2x)^3) = 3$.

1.7 $\omega = \frac{-1 + \sqrt{3}i}{2} : \omega^2 = \frac{-1 - \sqrt{3}i}{2} = \overline{\omega} = \frac{1}{\omega}$ (the square equals the conjugate equals the reciprocal!). That also implies that $\omega^3 = \omega\omega^2 = \omega\overline{\omega}$, which is 1, and likewise for $\overline{\omega}^3$.

1.8 Notice that $\zeta = -\overline{\omega}$, so $\zeta^3 = -1$ and $\zeta^6 = 1$, so $\zeta^{428347} = \zeta$.

1.11 Quaternions:

(a) $\mathbf{ijk} = -1$, so $\mathbf{ijk}^2 = -\mathbf{k}$, and thus $\mathbf{ij} = \mathbf{k}$ (since $\mathbf{k}^2 = -1$). This also leads us to $\mathbf{ik} = \mathbf{i}(\mathbf{ij}) = -\mathbf{j}$. The rest of the table can be filled out similarly.

$\cdot$	1	**i**	**j**	**k**
1	1	**i**	**j**	**k**
i	**i**	-1	**k**	**-j**
j	**j**	**-k**	-1	**i**
k	**k**	**j**	**-i**	-1

(b) This multiplication is not commutative since $\mathbf{ij} \neq \mathbf{ji}$. It is **anti-commutative** since $xy = -yx$ for $y \neq x$.

(c) i. $(5+3\mathbf{j}+3\mathbf{k})-(3+2\mathbf{i}-4\mathbf{j}+\mathbf{k}) = 2-2\mathbf{i}+7\mathbf{j}+2\mathbf{k}$.
ii. $(2\mathbf{j}+3\mathbf{k})\cdot(5+\mathbf{i}-2\mathbf{k}) = 10\mathbf{j}+2\mathbf{ji}-4\mathbf{jk}+15\mathbf{k}+3\mathbf{ki}-6\mathbf{k}^2 = 6-4\mathbf{i}+13\mathbf{j}+13\mathbf{k}$.

1.17 Let $z = a+bi, w = c+di$ for some $a,b,c,d \in \mathbb{R}$. Then

$$\begin{aligned}\overline{z+w} &= \overline{(a+bi)+(c+di)} \\ &= \overline{(a+c)+(b+d)i} \\ &= (a+c)-(b+di) \\ &= (a-bi)+(c-di) \\ &= \overline{z}+\overline{w}.\end{aligned}$$

$$\begin{aligned}\overline{zw} &= \overline{(a+bi)(c+di)} \\ &= \overline{(ac-bd)+(ad+bc)i} \\ &= (ac-bd)-(ad+bc)i.\end{aligned}$$

On the other hand,

$$\begin{aligned}\overline{z}\cdot\overline{w} &= \overline{(a+bi)}\cdot\overline{(c+di)} \\ &= (a-bi)(c-di) \\ &= (ac-bd)-(ad+bc)i.\end{aligned}$$

2.9 Let (x_0, y_0) be a rational solution to $ax^2+by^2 = c$ (i.e., a rational point on the ellipse). Let ℓ be the line through (x_0, y_0) with rational slope m. Then the second point of intersection (assuming ℓ is not tangent to the ellipse) is some point (x, y), and $y = m(x-x_0)+y_0$. Making this substitution (and re-writing slightly) gives $ax^2 + b(m(x-x_0)+y_0)^2 - c = 0$. Note that this is some quadratic equation in x with rational coefficients. Since we already know that $x = x_0$ is a solution, however the algebra works out on the right-hand side we can factor out $x - x_0 : (x-x_0)g(x) = 0$, where g is a linear polynomial in x. Since the coefficients in g are still rational, its root is also rational, as desired. (NB: This is an existence proof. If you want to know what the root actually is, you have to do the algebra!)

2.10 Determine whether the diagonals of an Euler brick can be a Pythagorean Triple.

They cannot. Suppose the side lengths of the Euler brick are a, b, and c. We would then require (without loss of generality) that $(a^2+b^2)+(b^2+c^2) = a^2+c^2$. This is only possible if $b = 0$, which isn't much of a brick.