MATH 356-01
Solutions to Homework Assignment 7

5.5 These are much easier to do if you graph them on a lattice.

(a) Since $N(4+2i) = 20$, there are 20 equivalence classes, and we want distinct ones with representatives of norm less than 20. Let S be the square with vertices at $(0,-2) \sim -2i, (4,0) \sim 4, (2,4) \sim 2+4i$, and $(-2,2) \sim -2+2i$. Then the coset representatives are the 17 lattice points inside the square along with $-2i, -1$, and $2-i$.

(b) This one is very similar to the previous one.

(c) Take the vertices at $-1, 2-i, 3+2i$, and $3i$. Use -1 and the 9 interior points as the 10 coset representatives.

(d) Vertices: $-3i, 5-i, 3+4i$, and $-2+2i$. Use the 28 interior points and $-3i$ as the coset representatives.

5.6 (a) Since $6-3i = 3(2-i), 6-3i \equiv 0 \pmod{2-i}$.

(b) $\dfrac{4+7i}{3+2i} = \dfrac{4+7i}{3+2i} \cdot \dfrac{3-2i}{3-2i} = \dfrac{26-13i}{13} = (2-i)$, so $4+7i \equiv 0 \pmod{3+2i}$.

(c) There are only two equivalence classes mod $1+i$: 0 and 1 (or 0 and i, if you prefer). Since $37 \not\equiv 14 \pmod 2$, $1+i$ does not divide $37-14i$, so it isn't 0 mod $1+i$. Therefore, it must be that $37-14i \equiv 1 \pmod{1+i}$.

5.11 (a) $3(5+3i) = 15+9i$, so $5+3i|15+9i$ is true.

(b) $N(5+13i) = 194$, while $N(5+14i) = 221$, and $194 \not| 221$, so $5+13i|5+14i$ is false.

(c) $i|7$ is true since i is a unit.

(d) $N(37-3i) = 1378 = 2 \cdot 13 \cdot 53 = (1+i)(1-i)(3+2i)(3-2i)(7+2i)(7-2i)$. On the other hand, $N(37-3i) = (37-3i)(37+3i)$, so half of the factors of $N(37-3i)$ go with $37-3i$ and half go with $37+3i$. Now $\dfrac{37-3i}{7-2i} \cdot \dfrac{7+2i}{7+2i} = \dfrac{265+53i}{53} = 5+i \in \mathbb{Z}[i]$, so $7-2i|37-3i$ is true.

(e) By the previous part, $7+2i|37-3i$ is false.

(f) $65 = 5 \cdot 13 = (2+i)(2-i)(3+2i)(3-2i)$, so $3+2i|65$ is true.

5.19 If $a+bi$ is a Gaussian prime, then either $N(a+bi)$ is prime or $a+bi \in \mathbb{Z}$ or $a+bi \in i\mathbb{Z}$. In the first case, the other elements are prime since they, too, have a prime norm. In the latter cases, a or b is a rational prime congruent to 3 mod 4 and the other is zero, so the Gaussian integers in question are associates of such a prime and therefore prime themselves.

5.21 Let γ and δ be gcds of α and β. By Exercise 5.27, $\gamma|\delta$ and $\delta|\gamma$, so there exist $\sigma, \tau \in \mathbb{Z}[i]$, such that $\delta = \gamma\sigma$ and $\gamma = \delta\tau = \gamma\sigma\tau$. since $\gamma \neq 0$, $\sigma\tau = 1$, so σ and τ are units. Therefore, γ and δ are associates.