MATH 356-01
Solutions to Homework Assignment 9

6.1 (a) if $x = \sqrt{7}$, then $x^2 - 7 = 0$. Since $\sqrt{7} \notin \mathbb{Q}$, the minimal polynomial is not linear, so it is $p(x) = x^2 - 7$.

(b) If $x = 2 - \sqrt{5}$, then $x - 2 = -\sqrt{5}$, so $x^2 - 4x + 4 = 5$, and $p(x) = x^2 - 4x - 1$ is the minimal polynomial (since the minimal polynomial isn't linear).

(c) If $x = \sqrt{6} + \sqrt{3}$, then $x - \sqrt{6} = \sqrt{3}$, so $x^2 - 2\sqrt{6}x + 6 = 3$, and thus $x^2 + 3 = 2\sqrt{6}x$. This gives $x^4 + 6x^2 + 9 = 24x$, so $p(x) = x^4 + 6x^2 - 24x + 9$. is the minimal polynomial (since the minimal polynomial isn't linear). This has no linear factors (with rational coefficients) since it has no rational roots by the Rational Root Test. It also has no quadratic factors. You can either try to factor it into quadratics and see that it fails, or you can take Abstract Algebra from me next year!(Or both, of course.)

(d)

$$\begin{aligned}
x &= \sqrt{10} - \sqrt[3]{5} \\
x - \sqrt{10} &= \sqrt[3]{5} \\
x^3 - 3\sqrt{10}x^2 + 30x - 10\sqrt{10} &= 5 \\
x^3 + 30x - 5 &= \sqrt{10}(3x^2 + 10) \\
x^6 + 60x^4 - 10x^3 + 900x^2 - 300x + 25 &= 10(9x^4 + 60x^2 + 100) \\
x^6 - 30x^4 - 10x^3 + 300x^2 - 300x - 975 &= 0.
\end{aligned}$$

Thus $p(x) = x^6 - 30x^4 - 10x^3 + 300x^2 - 300x - 975$.

To show this has least degree is not so easy...but take Abstract Algebra from me next year!

6.3 If $a + b\sqrt{d} = c + e\sqrt{d}$, then $a - c = (e - b)\sqrt{d}$. If $b \neq e$, then $\sqrt{d} = \dfrac{a - c}{e - b} \in \mathbb{Q}$, a contradiction since $\sqrt{d}$ is irrational for any non-square integer d. Therefore $b = e$, which then implies that $a = c$ as well.

6.11 Let $d, a, b, \in D$, where D is an integral domain. If $d|a$ and $d|b$, then $d|a + b$. More generally, $d|ax + by$ for any $x, y \in D$.

6.12 Let D be a Euclidean domain, and let $a, b \in D$, not both zero. If $d \in D$ is a gcd of a and b, then there exist $x, y \in D$ such that $d = ax + by$.