

# MATH 356 Number Theory

## Homework Assignment 11

### Keep:

1. Compute in the quaternions.
  - (a)  $(3 + 2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k})(2 - 3\mathbf{i} + \mathbf{k})$
  - (b)  $(2 + 6\mathbf{i} - \mathbf{j})(2 - 6\mathbf{i} + \mathbf{j})$
  - (c)  $4 - \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$  divided by  $1 - 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$  (find quotient and remainder).
2. Show that  $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ .
3. Show that  $\mathbf{ij} = \mathbf{k}$ ,  $\mathbf{jk} = \mathbf{i}$ , and  $\mathbf{ki} = \mathbf{j}$ .
4. Show that reversing the products in the previous parts changes the sign of the results.
5. Show that  $\mathbf{1}$ ,  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are linearly independent over  $\mathbb{R}$ .
6. Show that the Hurwitz integers form a ring with unity that has a multiplicative norm, a division algorithm, and no zero divisors.
7. Show that if  $\alpha$  is a quaternion represented by the matrix  $A$ , then  $\bar{\alpha}$  is represented by  $\bar{A}^T$ . [Recall that  $\overline{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$  and translate to matrices.]
8. Show that if  $q_1, q_2 \in Q$  (the quaternions), then  $\overline{q_1 q_2} = \bar{q}_2 \cdot \bar{q}_1$ .

### Turn in:

1. Let  $p$  be prime. Prove that if  $x^2 \equiv y^2 \pmod{p}$ , then  $x \equiv \pm y \pmod{p}$ .
2. Prove that  $Q$  (the quaternions) under matrix addition and multiplication is a non-commutative ring with unity and every non-zero element of  $Q$  has a multiplicative inverse that is also in  $Q$ .
3. Prove that the norm of a Hurwitz integer is an integer.

### Exam:

1. Prove that if  $r \in \mathbb{R}$  and  $q \in Q$  (the quaternions), then  $rq = qr$ .