# MATH 356

## The Diophantus Chord Method

Find **all** rational solutions to $x^2 + y^2 = 10$.

We have one pretty easily: $x = 3, y = 1$.

A line through $(3, 1)$ with rational slope will intersect the circle in another point (with the lone exception of the tangent line). That point will also be rational: $y = 1 + \frac{a}{b}(x - 3)$ and $x^2 + y^2 = 10$ implies that $x^2 + \left(1 + \frac{a}{b}(x - 3)\right)^2 = 10$.

Let's go to Python to solve!

```
from sympy import *
x,a,b=symbols('x,a,b')
q=solve(x**2+(1+a/b*(x-3))**2-10,x)[1]
```

That last line takes the second element of the list of solutions and calls it $q$. It won't print $q$ unless we ask it to, though.

```
print(q)
simplify(1+a/b*(q-3))
```

Python tells us $x = \dfrac{3a^2 - 2ab - 3b^2}{a^2 + b^2}$ (or 3, which we already have). If $a$ and $b$ are integers, then this is rational! Also, $y = \dfrac{-a^2 - 6ab + b^2}{a^2 + b^2}$. Conversely, if the second point is rational, so is the slope. Therefore, we have a (geometrical!) 1-1 correspondence between rational numbers and rational points on the circle.

Note: Because the equation is quadratic, `solve` returns two solutions in a list [3,messy solution]. Python indexes lists starting at 0, so the second solution has index 1. That's why on the `q=` line it has the [1] – it pulls out the element of the list with that index. (I solved first and saw that the solution I wanted was in that position, so then I went back and selected it.)

Example 2.3.3 (p. 17): Here is the Python code I used in class.

```
for x in range(2,100):
    for y in range(100):
        if 6*y**2 == x*(x+1)*(2*x+1):
            print(x,y)
total = 0
for i in range(1,25):
    total+=i**2
print(total)
```

(as a check!)

Note the arguments in range.