MATH 456-01

Solutions to Homework 1

Section 2.1
p. 30: 6, 12, 21, 22

6. Yes; it is true: Since $n|a-b$, $a-b=nx$ for some integer x. Since $k|n$, $n=ky$ for some integer y. Thus $a-b=kyx$, so $k|a-b$, and $a \equiv b (\text{mod } n)$.

12. Suppose $[p]=[0],[2]$, or $[4]$ mod 6. Then p must be even and, since $p \geq 5$, therefore not prime. If $[p]=[3]$, then $3|p$, so $p \geq 5$ again implies p is not prime. This leaves only [1] and [5] as possible congruence classes mod 6 for primes.

21. (a) If $n=1$, then $10^n = 10 \equiv 1 (\text{mod } 9)$, so the statement holds. Now assume that the statement holds for some $n \geq 1$. Then $10^{n+1} = 10^n \cdot 10 \equiv 1 \cdot 1 (\text{mod } n) \equiv 1 (\text{mod } n)$, as desired.

    (b) Let $n = a_k a_{k-1} \ldots a_2 a_1 a_0$ be the base-10 representation of n. That is, a_0 is the 1's digit of n, a_1 is the 10's digit, etc. This means that

$$\begin{aligned} n &= 10^k a_k + 10^{k-1} a_{k-1} + \ldots + 10^2 a_2 + 10 a_1 + a_0 \\ &\equiv 1 \cdot a_k + 1 \cdot a_{k-1} + \ldots + 1 \cdot a_2 + 1 \cdot a_1 + 1 \cdot a_0 \pmod 9 \\ &\equiv a_k + a_{k-1} + \ldots + a_2 + a_1 + a_1 \pmod 9, \end{aligned}$$

    as desired. (The second step uses part (a).)

22. (a) Take $n=6, a=2, b=3$, and $c=0$. We see $2 \cdot 3 \equiv 2 \cdot 0 \pmod 6$, but $3 \not\equiv 0 \pmod 6$.

    (b) Now suppose $\gcd(a,n)=1$. Then there exist integers x and y such that $xa+yn=1$, so $xa \equiv 1 \pmod n$. Now $ab \equiv ac \pmod n \implies xab \equiv xac \pmod n \implies b \equiv c \pmod n$.