MATH 456-01

Solutions to Homework 4

Section 3.1
p. 29: 1, 5, 13, 20, 29, 30, 35, 36, 42

1. (a) Closure fails: $1+3=4$, but $4 \notin S$.
   (b) Only 0 has an additive inverse.

5. I will apply the two-step subring test in each case.
   (a) Let $A = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix} \in M(\mathbb{R})$ with $r, s \in \mathbb{Q}$. Then $A - B = \begin{bmatrix} 0 & r-s \\ 0 & 0 \end{bmatrix}$ is also such a matrix since $r - s \in \mathbb{Q}$ and $AB = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is, too, since $0 \in \mathbb{Q}$. Thus, this is a subring. There is no identity since every product gives the zero matrix.
   (b) Certainly if A and B are two such matrices, so is $A - B$ since the 2,1-entry remains a zero and all other entries of the sum remain integers. If $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$, then $AB = \begin{bmatrix} ad & ae+bf \\ 0 & cf \end{bmatrix}$, which is another such matrix. Therefore this is a subring of $M(\mathbb{R})$. It has the identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
   (c) A quick check confirms that if A and B are two such matrices, then so are $A - B$ and AB. There is no identity: if $A = \begin{bmatrix} a & a \\ b & b \end{bmatrix}$, then $\begin{bmatrix} a & a \\ b & b \end{bmatrix}\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a(r+s) & a(t+u) \\ b(r+s) & b(t+u) \end{bmatrix}$. Thus we need $r + s = 1$ and $t + u = 1$. On the other hand, $\begin{bmatrix} r & s \\ t & u \end{bmatrix}\begin{bmatrix} a & a \\ b & b \end{bmatrix} = \begin{bmatrix} ra+sb & ra+sb \\ ta+ub & ta+ub \end{bmatrix}$. Therefore, $ra + sb = a$ and $ta + ub = b$. In order for the first equation to work, we need $s = 0$ (or changing b changes the sum), and for the second to work, we need $t = 0$. Thus $r = 1$ and $u = 1$, giving us the usual identity matrix (which is not in the subring). You can verify that this set does pass the two-step subring test.
   (d) This is a subring with unity (from $a = 1$).
   (e) This is also a subring with unity (use $a = 1$ again).

13. Let $z_1 = a + b\sqrt{2}, z_2 = c + d\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$. Then $z_1 - z_2 = (a-c) + (b-d)\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ and $z_1 z_2 = (ac + 2bd) + (ad + bd)\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$, so $\mathbb{Z}[\sqrt{2}]$ is a ring by the two-step subring test.

20. Notice that $R = \{3k | k \in \mathbb{Z}_{18}$. If $a, b \in R$, then $a = 3k, b = 3l$ for some $a, b \in \mathbb{Z}_{18}$. Now $a - b = 3(k - l) \in R$ and $ab = 3(3kl) \in R$, so R is a ring by the two-step subring test. It does not have an identity.

29. To complete the multiplication table, we rely on the distributive law: $s \cdot t = s(s+s) = s\cdot s + s\cdot s = t + t = s$. Similarly, $t \cdot s = (s+s)\cdot s = s$ and $t \cdot t = (s+s)\cdot(s+s) = s\cdot s + s\cdot s + s\cdot s + s\cdot s = t+t+t+t = s+s = t$. The multiplication table is thus

$\cdot$	r	s	t
r	r	r	r
s	r	t	s
t	r	s	t

30. $xy = x(x+x) = xx + xx = y + y = w$. $xz = x(x+y) = xx + xy = y + w = y$. $yx = (x+x)x = xx + xx = y + y = w$. $yz = y(x+y) = yx + yy = w + w = w$. $zx = (y+x)x = yx + xx = w + y = y$. The table is

$\cdot$	w	x	y	z
w	w	w	w	w
x	w	y	w	y
y	w	w	w	w
z	w	y	w	y

35. (a) $R \times S$ is not an integral domain: if $R = \mathbb{Z}_2$ and $S = \mathbb{Z}_2$, then $(0,1) \cdot (1,0) = (0,0)$ even though neither $(0,1)$ nor $(1,0)$ is zero in $R \times S$.

(b) This is also false. Using the same example as in (a), we see that $(1,1)$ is the multiplicative identity. However, $(1,0) \cdot (x,y) = (x,0) \neq (1,1)$ for any y, so $(1,0)$ has no multiplicative inverse.

36. Let $x \in \mathbb{R}$. If $x \leq 2$, then $f(x)g(x) = 0(2-x) = 0$. If $x > 2$, then $f(x)g(x) = (x-2) \cdot 0 = 0$. Thus $f(x)g(x) = 0$ for all $x \in \mathbb{R}$, so $fg = 0_T$.

42. (a) Since $b \neq 0$, $bx = 1_R$ has a solution. Now $bb = b \implies (bb)x = bx \implies b(bx) = bx \implies b \cdot 1_R = 1_R$, so $b = 1_R$.

(b) Applying the hint, we see that $(ua)(ua) = u(au)a = u \cdot 1_R \cdot a = ua$. Thus, by part (a), $ua = 1_R$.