MATH 456-01

Solutions to Homework 5

Section 3.2
p. 62: 1, 3, 13, 15, 17, (23), 28, 29

1. (a) $(a+b)(a-b) = aa + a(-b) + ba + b(-b) = a^2 - ab + ba - b^2$.

   (b) $(a+b)^3 = (a+b)(a+b)(a+b) = (aa+ab+ba+bb)(a+b) = aaa+aab+aba+abb+baa+bab+bba+bbb = a^3 + a^2b + aba + ab^2 + ba^2 + bab + b^2a + b^3$.

   (c) (a) becomes $a^2 - b^2$ and (b) becomes $a^3 + 3a^2b + 3ab^2 + b^3$.

3. (a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are easily seen to be idempotent. A little experimentation will yield $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ as well.

   (b) In $\mathbb{Z}_{12}$, $0^2 = 0, 1^2 = 1, 4^2 = 4$, and $9^2 = 9$.

13. (a) $S \cap T$ is a subring of R. If $a, b \in S \cap T$, then $a, b \in S \implies a - b, ab \in S$ and $a, b \in T \implies a - b, ab \in T$. Thus $S \cap T$ passes the two-step subring test.

    (b) $S \cap T$ need not be a subring of R : in $\mathbb{Z}_6$, for example, $S = \{0, 2, 4\}$ and $T = \{0, 3\}$ are both subrings (check), but $S \cup T = \{0, 2, 3, 4\}$ is not closed under addition.

15. (a) To prove that ab is a unit, we will just show that it has an inverse: $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = a(1_R a^{-1}) = aa^{-1} = 1_R$. Similarly, $(b^{-1}a^{-1})(ab) = 1_R$.

    (b) If the ring is commutative, then $a^{-1}b^{-1} = b^{-1}a^{-1}$, so $a^{-1}b^{-1}$ would be the multiplicative inverse of ab. To find the example we seek, we need to look in a non-commutative ring like $M_{2\times 2}(\mathbb{Z})$. Consider $a = \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}, b = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. $a^{-1} = \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix}$ and $b^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. We get $ab = \begin{bmatrix} 6 & 11 \\ 7 & 13 \end{bmatrix}$, so $(ab)^{-1} = \begin{bmatrix} 13 & -11 \\ -7 & 6 \end{bmatrix}$. On the other hand, $a^{-1}b^{-1} = \begin{bmatrix} 14 & -23 \\ -3 & 5 \end{bmatrix} \neq (ab)^{-1}$.

17. It is not possible. Suppose x is a unit in a ring R with identity 1_R and $xr = 0$ for some $r \in R$. Then there is an element $u \in R$ such that $ux = 1_R$, so $uxr = u \cdot 0 \implies 1_R r = 0 \implies r = 0$. Thus x fails condition (2) of the definition of a zero divisor.

28. We can start by inserting the products with 0_R and 1_R:

$\cdot$	0_R	1_R	a	b
0_R	0_R	0_R	0_R	0_R
1_R	0_R	1_R	a	b
a	0_R	a		
b	0_R	b		

By Theorem 3.8, $ax = a$ has a unique solution, so a cannot appear again in the a row of the table. Similarly, there cannot be another b in the b column of the table (since the solution of $yb = b$ is unique). Thus $ab = 1_R$ and the rest of table practically fills itself in!

$\cdot$	0_R	1_R	a	b
0_R	0_R	0_R	0_R	0_R
1_R	0_R	1_R	a	b
a	0_R	a	b	1_R
b	0_R	b	1_R	a

29. We already know that if R is an integral domain, then cancellation holds in R. For the converse, suppose that cancellation holds in R and that there exist $a, b \in R$ such that $a \neq 0$ but $ab = 0$. Since $a \cdot 0 = 0$, we have $ab = a \cdot 0$, so cancellation implies $b = 0$. Therefore, R is an integral domain.