MATH 456-01

Solutions to Homework 6

Section 3.3
p. 80: 1, 3, 4, 11, 12, 29, 31, 34, 35

1. See back of book. The isomorphism has $[0]_6 \to ([0]_2, [0]_3), [1]_6 \to ([1]_2, [1]_3), [2]_6 \to ([0]_2, [2]_3), [3]_6 \to ([1]_2, [0]_3), [4]_6 \to ([0]_2, [1]_3), [5]_6 \to ([1]_2, [2]_3)$.

3. Since f is clearly a bijective function ((a,a) is hit by a and $f(a) = f(b) \implies (a,a) = (b,b) \implies a = b$), we need only show that it is also operation-preserving. Now $f(a+b) = (a+b, a+b) = (a,a) + (b,b) = f(a) + f(b)$ and $f(ab) = (ab, ab) = (a,a)(b,b) = f(a)f(b)$, so f is an isomorphism.

4. Since $\overline{3} \cdot \overline{3} = \overline{4}$ in $\mathbb{Z}_5$, we must have $f(\overline{3} \cdot \overline{3}) = f(\overline{4}) = 8$. On the other hand, $f(\overline{3}) \cdot (\overline{3}) = 6 \cdot 6 = 6$ in $\mathbb{Z}_6$, so multiplication is not preserved. Therefore, f is not an isomorphism.

12. (a) This is not a homomorphism: $f(1 \cdot 1) = f(1) = -1$, but $f(1)f(1) = (-1)(-1) = 1$.
    (b) This is an isomorphism (and hence a homomorphism, too) since $1 = -1$ and $0 = 0$ in $\mathbb{Z}_2$. That is, for all $x \in \mathbb{Z}_2$, $x = -x$, so f is the identity function on $\mathbb{Z}_2$.
    (c) This is not a homomorphism: $g(0+0) = g(0) = \dfrac{1}{0^2+1} = 1$, but $g(0) + g(0) = 2$.
    (d) This is not a homomorphism: $h(1 \cdot 1) = h(1) = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$, but $h(1)h(1) = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$.
    (e) This is a homomorphism. We must show first that it is well-defined: if $[a]_{12} = [b]_{12}$, then $a \equiv b \pmod{()12}$, so $12|a-b$. Therefore $4|a-b$, so $[a]_4 = [b]_4$ and $f([a]_{12}) = f([b]_{12})$. To show that f is operation-preserving, let $[a]_{12}, [b]_{12} \in \mathbb{Z}_{12}$. Then $f([a]_{12} + [b]_{12}) = f([a+b]_{12}) = [a+b]_4 = [a]_4 + [b]_4 = f([a]_{12}) + f([b]_{12})$. Also, $f([a]_{12}[b]_{12}) = f([ab]_{12}) = [ab]_4 = [a]_4[b]_4 = f([a]_{12})f([b]_{12})$.

29. Let $a, b \in S$. Then there exist $x, y \in R$ such that $f(x) = a$ and $f(y) = b$ since f is a surjection. This implies that $g(a) = x$ and $g(y) = b$ since g is the inverse of f. Now $g(a+b) = g(f(x) + f(y)) = g(f(x+y)) = x + y = g(a) + g(b)$ and $g(ab) = g(f(x)f(y)) = g(f(xy)) = xy = g(a)g(b)$, using the fact that f is operation-preserving.

34. (a) This is preserved. If a is a zero divisor, then there exists $b \in R$ such that $b \neq 0$ and $ab = 0$. Now $f(a)f(b) = f(ab) = f(0_R) = 0_S$, and $f(a), f(b) \neq 0_S$ since f is one-to-one. Therefore, $f(a)$ is also a zero divisor.
    (b) This is preserved. If e is an idempotent, then $[f(e)]^2 = f(e)f(e) = f(e^2) = f(e)$.
    (c) This is preserved. The image of a ring is a ring, so we need only show that S is a commutative ring with identity and no zero divisors. Commutativity is certainly preserved: if $x, y \in S$, then, since f is surjective, there exist $a, b \in R$ such that $f(a) = x$ and $f(b) = y$. Now $xy = f(a)f(b) = f(ab) = f(ba) = f(b)f(a) = yx$. We saw in class that if f is surjective and R has a unity, then S has a unity and $f(1_R) = 1_S$. Finally, if $xy = 0_S$, then $f(a)f(b) = 0_S$ (using the notation from the commutativity argument), so $f(ab) = 0_S$. Thus $ab = 0_R$, so $a = 0_R$ or $b = 0_R$, as desired.

35. (a) $\mathbb{Z}$ has 2 units; E has none.
    (b) The first ring is commutative; the second is not.
    (c) There is no bijection between these sets.
    (d) There is no bijection between $\mathbb{Q}$ and $\mathbb{R}$.
    (e) $\mathbb{Z} \times \mathbb{Z}_2$ is not an integral domain, but $\mathbb{Z}$ is.
    (f) $\mathbb{Z}_4 \times \mathbb{Z}_4$ has 4 solutions to the equation $2x = 0$; $\mathbb{Z}_{16}$ has only 2.