MATH 456-01

Solutions to Homework 8

Section 4.2
p. 99: 3, 5ac, 6ac, 9, 15

3. From the proof of theorem 4.5, the gcd of $x + a$ and $x + b$ is the linear combination of $x + a$ and $x + b$ that is (a) monic and (b) of smallest (finite) degree. Notice that $(x + a) - (x + b) = a - b$ has degree 0. Since there is no smaller finite degree, the gcd must be a constant polynomial. Since the only monic constant polynomial is 1_F, $\gcd(x + a, x + b) = 1$ and the two are relatively prime.

5. (a)

$$\begin{aligned} x^4 - x^3 - x^2 + 1 &= (x^3 - 1)(x - 1) - x^2 + x \\ x^3 - 1 &= (-x^2 + x)(-x - 1) + x - 1 \\ -x^2 + x &= (x - 1)(-x). \end{aligned}$$

   Thus $\gcd(x^4 - x^3 - x^2 + 1, x^3 - 1) = x - 1$.

   (c)

$$\begin{aligned} x^4 + 3x^3 + 2x + 4 &= (x^2 - 1)(x^2 + 3x + 1) + 5x - 5 \\ &= (x^2 - 1)(x^2 + 3x + 1). \end{aligned}$$

   Thus $\gcd(x^4 + 3x^3 + 2x + 4, x^2 - 1) = x^2 - 1$ in $\mathbb{Z}_5[x]$.

6. (a)

$$\begin{aligned} x - 1 &= (x^3 - 1) + (-x^2 + x)(x + 1) \\ &= (x^3 - 1) + [(x^4 - x^3 - x^2 + 1) - (x^3 - 1)(x - 1)](x + 1) \\ &= (x^3 - 1)[1 - (x^2 - 1)] + (x^4 - x^3 - x^2 + 1)(x + 1) \\ &= (x^3 - 1)[2 - x^2] + (x^4 - x^3 - x^2 + 1)(x + 1) \end{aligned}$$

   c This is the easy one: $x^2 - 1 = 0(x^4 + 3x^3 + 2x + 4) + 1(x^2 - 1)$.

9. It must be the case that $f(x)$ is a constant polynomial: if $f(x)$ is nonconstant, then f has an irreducible factor of degree at least 1. Since every polynomial divides 0_F, the gcd would also have degree at least 1, a contradiction.

15. Write $f(x) = h(x)q(x)$. Since $f(x)$ and $g(x)$ are relatively prime, there are polynomials $u(x)$ and $v(x)$ such that

$$f(x)u(x) + g(x)v(x) = 1.$$

   Thus $h(x)q(x)u(x) + g(x)v(x) = 1$, so $\gcd(h(x), g(x)) = 1$ as well (see the argument in number 3).