MATH 456-01

Solutions to Homework 9

Section 4.3
p. 3, 6, 9, 10, 12, 23

3. (a) The units in $\mathbb{Z}_5[x]$ are $1, 2, 3, 4$, so the associates are $x^2+x+1, 2x^2+2x+2, 3x^2+3x+3$, and $4x^2+4x+4$.

   (b) The units in $\mathbb{Z}_7[x]$ are $1, 2, 3, 4, 5, 6$ so the associates are $3x+2, 6x+4, 2x+6, 5x+1, x+3$, and $4x+5$.

6. Factoring as suggested, we get $x^2+1 = (ax+b)(cx+d) = acx^2+(ad+bc)x+bd$, so $ac = 1, ad+bc = 0$, and $bd = 1$. Thus, none of a, b, c, d can be zero, so $c = \dfrac{1}{a}, d = \dfrac{1}{b}$, and $ad+bc = \dfrac{a}{b}+\dfrac{b}{a} = 0$. This gives $\dfrac{a^2+b^2}{ab} = 0$, so $a^2+b^2 = 0$. This is impossible unless $a = b = 0$, which is not the case. Therefore, x^2+1 is irreducible in $\mathbb{Q}[x]$.

9. (a) For a polynomial in $\mathbb{Z}_2[x]$ to be irreducible, it must be the case that it cannot be factored into linear factors. The only linear polynomials are x and $x+1$, so the reducible degree-two polynomials are $x^2, x(x+1) = x^2+x$, and $(x+1)^2 = x^2+1$. This leaves only x^2+x+1 as an irreducible polynomial.

   (b) A reducible polynomial of degree 3 must factor into 3 linear factors or one linear factor and one quadratic factor. The former gives $x^3, x^2(x+1) = x^3+x^2, x(x+1)^2 = x^3+x$, and $(x+1)^3 = x^3+x^2+x+1$. The latter gives $x(x^2+x+1) = x^3+x^2+x$ and $(x+1)(x^2+x+1) = x^3+1$. This leaves only x^3+x+1 and x^3+x^2+1 as irreducible.

   (c) The linear polynomials in $\mathbb{Z}_3[x]$ are $x, 2x, x+1, x+2, 2x+1$, and $2x+2$. There are a total of 18 polynomials of degree two in this ring (2 choices for the leading coefficient and 3 for each of the other two). Since x cannot be a factor, the constant term cannot be 0. (That leaves us with only 12 options.) Also, we may as well only look for monic irreducibles since each non-monic irreducible is twice some monic irreducible. [All associates of an irreducible are also irreducible.] Now there are only six polynomials to consider: $x^2+1, x^2+2, x^2+x+1, x^2+x+2, x^2+2x+1, x^2+2x+2$.

   A quick check shows us that $x^2+2 = (x+1)(x+2), x^2+x+1 = (x+2)^2$, and $x^2+2x+1 = (x+1)^2$. [Consider only products of monic linear factors: if one is monic and the other is not, the product won't be either. If neither is monic, then both are negatives (mod 3) of monic polynomials, so the product will match one of the products found with monic polynomials.] The others are irreducible: x^2+1, x^2+x+2, and x^2+2x+2. Multiplying each by two gives the rest of the irreducible polynomials: $2x^2+2, 2x^2+2x+1$, and $2x^2+x+1$.

10. (a) $x^2-3 = (x-\sqrt{3})(x+\sqrt{3})$ factors in $\mathbb{R}[x]$, but it is irreducible in $\mathbb{Q}[x]$.

   (b) $x^2+x-2 = (x+2)(x-1)$ factors in both rings.

12. $x^4 - 4 = (x^2 - 2)(x^2 + 2)$ is the factorization in $\mathbb{Q}[x]$. We may factor a little further in $\mathbb{R}[x] : (x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)$. We may factor further yet in $\mathbb{C}[x] : (x - \sqrt{2})(x + \sqrt{2})(x - \sqrt{2}i)(x + \sqrt{2}i)$.

23. (a) Suppose that $x^2 + 2 = (x + a)(x + b)$ in $\mathbb{Z}_5[x]$. (Since $x^2 + 2$ is monic, we may reason as in number 21 that our linear factors may be assumed to be monic.) We have $x^2 + 2 = x^2 + (a + b)x + ab$, so $b = -a$ and $ab = 2$. Thus $-a^2 = 2$, or $a^2 = -2 = 3$ (in $\mathbb{Z}_5$). Now $0^2 = 0, 1^2 = 1, 2^2 = 4, 3^2 = 4$, and $4^2 = 1$, so there is no solution to $a^2 = 3$ in $\mathbb{Z}_5$. Therefore, $x^2 + 2$ is irreducible in $\mathbb{Z}_5[x]$.

    (b) $x^4 - 4 = (x^2 - 2)(x^2 + 2)$. Arguing as in (a), we would need a, b such that $b = -a$ and $ab = -2$, which becomes $a^2 = 2$. This also has no solution, so $x^2 - 2$ is also irreducible, and we are done.