MATH 456-01

Solutions to Homework 10

Section 4.4
p. 1, 2ad, 3, 8abdef, 10, 17

1. (a) $f(x) = x^2 + x$ induces the zero function on $\mathbb{Z}_2 : f(0) = 0$ and $f(1) = 0$.
   (b) $f(x) = x^3 + 2x$ induces the zero function on $\mathbb{Z}_3$: $f(0) = 0, f(1) = 0$, and $f(2) = 0$.
2. (a) $f(1) = 2$, so the remainder is 2.
   (d) $f(3) = 2(243) - 3(81) + 27 + 6 + 3 = 2(3) - 3(1) + 2 + 1 + 3 = 4$ in $\mathbb{Z}_5$.
3. (a) $f(-2) = -8 - 12 + 8 - 12 = 24$, so $x + 2$ is not a factor.
   (b) $f(1/2) = 1/8 + 1/8 + 1/2 - 3/4 = 0$, so $x - 1/2$ is a factor of $f(x)$.
   (c) $f(-2) = 3(-32) + 4(16) + 2(8) - 4 + 4 + 1 = 0$ in $\mathbb{Z}_5$ so $x + 2$ is a factor of $f(x)$.
   (d) $h(3) = (27)(27) - 27 + 3 - 5 = 1 + 1 + 3 - 5 = 0$ in $\mathbb{Z}_7$, so $x - 3$ is a factor of $f(x)$.
8. (a) $x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7})$, so $x^2 - 7$ is not irreducible in $\mathbb{R}[x]$.
   (b) Since $\pm\sqrt{7} \notin \mathbb{Q}$, $x^2 - 7$ is irreducible in $\mathbb{Q}[x]$.
   (d) Substituting $x = 0, 1, 2, 3, 4$ in turn never gives zero, so this polynomial is irreducible over $\mathbb{Z}_5$.
   (e) Since $4^3 - 9 = 64 - 9 = 55 = 0$ in $\mathbb{Z}_{11}$, $x^3 - 9$ is not irreducible over $\mathbb{Z}_{11}$. It has a factor of $x - 4$.
   (f) Since $1^4 + 1^2 + 1 = 0$ in $\mathbb{Z}_3$, this polynomial is not irreducible over $\mathbb{Z}_3$. It has a factor of $x - 1$.
10. Try $p = 13$. Then $5^2 + 1 = 26 = 0$ in $\mathbb{Z}_{13}$, so $x^2 + 1$ has a factor of $x - 5$. [For you number theory folks: we are looking for a prime p such that -1 is a quadratic residue (mod p). This happens if and only if $p \equiv 1 \pmod 4$. See page 99 of your Number Theory text (Theorem 5.5).]
17. We've already seen that this is true of $x^2 + x$. It does not contradict the corollary because $\mathbb{Z}_6$ is not a field.