MATH 456-01

Solutions to Homework 11

Section 4.5
p. 113: 1-5, 7, 11, 12, 13, 18

1. (a) The possible roots are $\pm 1, \pm 2$. $x = -1$ is a root, so we have $-x^4+x^3+x^2+x+2 = (x+1)(-x^3+2x^2-x+2)$. We can factor this by grouping: $-x^3+2x^2-x+2 = -x^2(x-2)-(x-2) = -(x-2)(x^2+1)$. The complete factorization is thus $-(x+1)(x-2)(x^2+1)$.

   (b) Firstly, $x^5+4x^4+x^3-x^2 = x^2(x^3+4x^2+x-1)$. The possible rational roots of the second factor are ± 1. Since neither of these is a root, we have a complete factorization already.

   (c) Begin by factoring out x^2: $3x^5+2x^4-7x^3+2x^2 = x^2(3x^3+2x^2-7x+2)$. The possible rational roots are $\pm 1, \pm 1/3, \pm 2, \pm 2/3$. We can see that 1 is a root, so $x-1$ is a factor. We get $3x^5+2x^4-7x^3+2x^2 = x^2(x-1)(3x^2+5x-2)$. Our quadratic factor has the same potential rational roots. A quick check shows that -2 is a root, so $x+2$ is a factor, and we have $3x^5+2x^4-7x^3+2x^2 = x^2(x-1)(x+2)(3x-1)$.

   (d) The possible rational roots are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 1/2, \pm 3/2$. $x = -1$ is a root, so $x+1$ is a factor, giving us $2x^4-5x^3+3x^2+4x-6 = (x+1)(2x^3-7x^2+10x-6)$. This has the same potential rational roots. Notice that the signs of the cubic factor alternate. That means that we can't have a negative root, as there will never be cancellation. Some calculation yields that $x = 3/2$ is a root, so $x-3/2$ is a factor. We get $2x^4-5x^3+3x^2+4x-6 = (x+1)(x-3/2)(2x^2-4x+4) = (x+1)(2x-3)(x^2-2x+2)$. The quadratic factor is irreducible by Eisenstein's criterion.

   (e) Since all coefficients here have the same sign, this cannot have any positive roots. Therefore, we need only consider $-1, -3, -1/2, -3/2$. $x = -3$ is a root, giving us $2x^4+7x^3+5x^2+7x+3 = (x+3)(2x^3+x^2+2x+1) = (x+3)(2x+1)(x^2+1)$.

   (f) Our possible factors are $\pm 1, \pm 7, \pm 1/2, \pm 7/2, \pm 1/3, \pm 7/3, \pm 1/6, \pm 7/6$. I'm really hoping for an integer root here. No such luck. Some heinous computation shows that $x = -1/3$ is a root, so $x+1/3$ is a factor. We get $6x^4-31x^3+25x^2+33x+7 = (3x+1)(2x^3-11x^2+12x+7)$. The cubic has slightly fewer possible rational roots than did the original quartic: $\pm 1, \pm 7, \pm 1/2, \pm 7/2$. $x = 7/2$ is also a root, so $x-7/2$ is a factor. We now have $6x^4-31x^3+25x^2+33x+7 = (3x+1)(2x^3-11x^2+12x+7) = (3x+1)(2x-7)(x^2-2x-1)$.

2. The roots of x^2-p are $\pm\sqrt{p}$. However, by Eisenstein's criterion, x^2-p is irreducible over $\mathbb{Q}$, so $\pm\sqrt{p} \notin \mathbb{Q}$ by the Factor Theorem.

3. By the rational root test, any rational root of such a polynomial must be of the form $\pm\frac{r}{1} = \pm r$, where r is a factor of the constant term.

4. The only possible rational roots of either polynomial are ± 1, and neither is a root of either polynomial. Thus, for both (a) and (b), any factors must be quadratic. In addition, since we may factor with polynomials in $\mathbb{Z}[x]$, we may assume both quadratic factors are monic.

   (a) If $x^4 + 2x^3 + x + 1 = (x^2 + ax + b)(x^2 + cx + d)$, then we have $a + c = 2, b + d + ac = 0, ad + bc = 1$, and $bd = 1$. Thus $b = d = 1$ or $b = d = -1$. In either case, we get $a + c = 2$ and $ac = \pm 2$, which is not possible with integers.

   (b) Here we have $x^4 - 2x^2 + 8x + 1 = (x^2 + ax + b)(x^2 + cx + d)$, which gives $a + c = 0, b + d + ac = -2, ad + bc = 8$, and $bd = 1$. This gives $c = -a$, so $b + d = a^2 - 2$, $a(d - b) = 8$, and $b = d = 1$ or $b = d = -1$. In either case, $d - b = 0$, so $a(d - b) \neq 8$.

5. (a) Since $2|(-4)$ and $2|22$, but $4 \nmid 22$, this is irreducible by Eisenstein's criterion.

   (b) Here, $5|10, 5|(-15)$, and $5|25$, but $5 \nmid (-7)$ and $25 \nmid 10$, so this is irreducible by Eisenstein's criterion.

   (c) We may use either $p = 2$ or $p = 3$. Both divide every coefficient except the leading coefficient, but the square of neither divides -6.

7. (a) Neither $p = 2$ nor $p = 3$ will help since both eliminate the constant term. I will try $p = 5$. The polynomial becomes $2x^3 + x^2 + 4x + 1$. Since none of $0, 1, 2, 3, 4$ is a root of this, it is irreducible in $\mathbb{Z}_5[x]$ and hence in $\mathbb{Q}[x]$.

   (b) In $\mathbb{Z}_2[x]$, this becomes $x^4 + x + 1$. Since neither 0 nor 1 is a root of this, it is irreducible in $\mathbb{Z}_2[x]$ and hence in $\mathbb{Q}[x]$.

11. Since $91 = 7 \cdot 13$, we have $7|91, 7^2 \nmid 91$, and $7 \nmid 30$. Therefore, by Eisenstein's criterion, $30x^n - 91$ is irreducible over $\mathbb{Q}$. Thus $30x^n - 91$ can have no rational roots.

12. If $f(x)$ is reducible, then $f(x) = g(x)h(x)$ for some $g(x), h(x) \in F[x]$. Thus $f(x + c) = g(x + c)h(x + c)$, so $f(x + c)$ is also reducible over F.

13. $f(x + 1) = (x + 1)^4 + 4(x + 1) + 1 = x^4 + 4x^3 + 6x^2 + 8x + 6$. Since 2 divides every coefficient except the leading coefficient and $4 \nmid 6$, Eisenstein's criterion implies that $f(x + 1)$ is irreducible. By the previous exercise, so is $f(x)$.

18. (a) The only possible rational roots are ± 1, and neither is a root. Therefore, $x^4 - x^2 + 1$ is irreducible over $\mathbb{Q}$.

    (b) The same argument applies here.

    (c) Since none of $\pm 1, \pm 5$ are roots, this is irreducible over $\mathbb{Q}$.

    (d) Reducing this mod 2 gives $x^5 + x^2 + 1$, which has no roots in $\mathbb{Z}_2$. Therefore, the given polynomial is also irreducible over $\mathbb{Q}$.