MATH 456-01

Solutions to Homework 12

Section 4.6
p. 117: 1-6, 8

1. (a) $1+2i$ is another root, and thus $(x-1-2i)(x-1+2i) = x^2-2x+5$ is a factor of $x^4-3x^2+x^2-7x-30$. We get $x^4-3x^2+x^2-7x-30 = (x^2-2x+5)(x^2-x-6)$. Since $x^2-x-6 = (x-3)(x+2)$, the other roots are -2 and 3.

   (b) $1-i$ is another root, so $(x-1-i)(x-1+i) = x^2-2x+2$ is a factor. We have $x^4-2x^3-x^2+6x-6 = (x^2-2x+2)(x^2-3)$, so the other roots are $\pm\sqrt{3}$.

   (c) $3-2i$ is another root, so $x^2-6x+13$ is a factor. We have $x^4-4x^3+3x^2+14x+26 = (x^2-6x+13)(x^2+2x+2)$, so the other roots are $-1 \pm i$.

2. (a) We must also have $3-i$ as a root, so $x-2$ and $x^2-6x+10$ are factors. The polynomial is $x^3-8x^2+22x-20$.

   (b) The degree must be at least 4 since we have four distinct roots. The quadratic factors are x^2-2x+2 and x^2+4. The polynomial is $x^4-2x^3+6x^2-8x+8$.

   (c) The degree here is three, and the factors are $x-3$ and $x^2-2x+17$. The polynomial is $x^3-5x^2+23x-51$.

3. (a) x^4-2 is irreducible over $\mathbb{Q}$ by Eisenstein's Criterion. It factors over $\mathbb{R}$ as $(x^2-\sqrt{2})(x^2+\sqrt{2}) = (x-\sqrt[4]{2})(x+\sqrt[4]{2})(x^2+\sqrt{2})$. Finally, it factors over $\mathbb{C}$ as $(x-\sqrt[4]{2})(x+\sqrt[4]{2})(x-i\sqrt[4]{2})(x+i\sqrt[4]{2})$.

   (b) $x^3+1 = (x+1)(x^2-x+1)$, and x^2-x+1 is irreducible over $\mathbb{Q}$ by the mod p test with $p=2$. x^2-x+1 is also irreducible over $\mathbb{R}$ since $(-1)^2-4(1)(1) = -3 < 0$. Our polynomial factors over $\mathbb{C}$ as $(x+1)\left(x-\dfrac{1+i\sqrt{3}}{2}\right)\left(x-\dfrac{1-i\sqrt{3}}{2}\right)$.

   (c) $x^3-x^2-5x+5 = (x-1)(x^2-5)$ is the factorization over $\mathbb{Q}$. Over both $\mathbb{R}$ and $\mathbb{C}$, the factorization is $(x-1)(x-\sqrt{5})(x+\sqrt{5})$.

4. The roots are $x = \dfrac{-1\pm\sqrt{1-4-4i}}{2} = \dfrac{-1\pm\sqrt{-4i-3}}{2}$. We need the square root of $-4i-3$, so suppose $(a+bi)^2 = -4i-3$. Then $a^2-b^2+2abi = -4i-3$. Thus $a^2-b^2=-3$ and $2ab=-4$, or $b^2-a^2=3$ and $ab=-2$. $b=2$ and $a=-1$ solves this, so we have roots $x = \dfrac{-1\pm(-1+2i)}{2}$. Therefore $x=-i$ or $x=-1+i$. The factorization is therefore $(x+i)(x+1-i)$.

5. Since non-real complex roots all appear in pairs, the polynomial will factor as a product of linear factors and irreducible quadratic factors. Since there are no repeated roots, each linear factor corresponds to a different root. Also, because the degree of the polynomial is odd, there must be an odd number of linear factors, and hence an odd number of real roots.

6. This is a standard exercise; follow the hint.

8. Not necessarily. The proof of Lemma 4.28 fails because conjugation changes not only the roots, but also the coefficients of the polynomial. In fact, i is a root, but $-i$ is not a root.