MATH 456-01

Solutions to Homework 14

Section 5.2
p. 134: 2, 5, 6, 7, 8, 11, 14

2. Remember here that $x^2 + 1 = 0$, or $x^2 = 2$.

$+$	$[0]$	$[1]$	$[2]$	$[x]$	$[x+1]$	$[x+2]$	$[2x]$	$[2x+1]$	$[2x+2]$
$[0]$	0	1	2	x	$x+1$	$x+2$	$2x$	$2x+1$	$2x+2$
$[1]$	1	2	0	$x+1$	$x+2$	x	$2x+1$	$2x+2$	$2x$
$[2]$	2	0	1	$x+2$	x	$x+1$	x	$2x$	$2x+1$
$[x]$	x	$x+1$	$x+2$	$2x$	$2x+1$	$2x+2$	0	1	2
$[x+1]$	$x+1$	$x+2$	x	$2x+1$	$2x+2$	$2x$	1	2	0
$[x+2]$	$x+2$	x	$x+1$	$2x+2$	$2x$	$2x+1$	2	0	1
$[2x]$	$2x$	$2x+1$	$2x+2$	0	1	2	x	$x+1$	$x+2$
$[2x+1]$	$2x+1$	$2x+2$	$2x$	1	2	0	$x+1$	$x+2$	x
$[2x+2]$	$2x+2$	$2x$	$2x+1$	2	0	1	$x+2$	x	$x+1$

$\cdot$	$[0]$	$[1]$	$[2]$	$[x]$	$[x+1]$	$[x+2]$	$[2x]$	$[2x+1]$	$[2x+2]$
$[0]$	0	0	0	0	0	0	0	0	0
$[1]$	0	1	2	x	$x+1$	$x+2$	$2x$	$2x+1$	$2x+1$
$[2]$	0	2	1	$2x$	$2x+2$	$2x+1$	x	$x+2$	$x+1$
$[x]$	0	x	$2x$	2	$x+2$	$2x+2$	1	$x+1$	$2x+1$
$[x+1]$	0	$x+1$	$2x+2$	$x+2$	$2x$	1	$2x+1$	2	x
$[x+2]$	0	$x+2$	$2x+1$	$2x+2$	1	x	$x+1$	$2x$	2
$[2x]$	0	$2x$	x	1	$2x+1$	$x+1$	2	$2x+2$	$x+2$
$[2x+1]$	0	$2x+1$	$x+2$	$x+1$	2	$2x$	$2x+2$	x	1
$[2x+2]$	0	$2x+2$	$x+1$	$2x+1$	x	2	$x+2$	1	$2x$

And if you don't think that was a major pain, you are mistaken!

For Exercises 5-8, we can choose linear (or constant) representatives for each equivalence class by Corollary 5.5 since each polynomial is quadratic. Notice that in each ring, we have $[ax+b]+[cx+d] = [(a+c)x+(b+d)]$, so we only need a rule for multiplication.

5. In this ring, $x^2 = -1$. We have $[ax+b][cx+d] = [acx^2 + (ad+bc)x + bd] = [(ad+bc)x + (bd-ac)]$.

6. In this ring, $x^2 = 2$, so we get $[ax+b][cx+d] = [acx^2 + (ad+bc)x + bd] = [(ad+bc)x + (2ac+bd)]$.

7. In this ring, $x^2 = 3$, so we get $[ax+b][cx+d] = [acx^2 + (ad+bc)x + bd] = [(ad+bc)x + (3ac+bd)]$.

8. In this ring, $x^2 = 0$, so we get $[ax+b][cx+d] = [acx^2 + (ad+bc)x + bd] = [(ad+bc)x + bd]$.

11. $\mathbb{Q}[x]/(x^2)$ is not a field; in fact, it is not even an integral domain: $[x]$ is a zero divisor! $[x][x] = [x^2] = [0]$.

14. (a) $[2x-3]$ is a unit because $[2x-3]$ is relatively prime to (x^2-2). Suppose its inverse is $[ax+b]$. Then $[2x-3][ax+b] = [(2b-3a)x + (4a-3b)]$ (using the formula from Exercise 6). Thus, we need $2b - 3a = 0$ and $4a - 3b = 1$. Solving gives $a = -2$ and $b = -3$, so $[2x-3]^{-1} = [-2x-3]$.

    (b) $[f(x)] = [x^2+x+1] = [(x^2+1)+x] = [x]$ in this ring. Since x is relatively prime to x^2+1 in $\mathbb{Z}_3[x]$, Theorem 5.9 implies that $[x]$ is a unit in $\mathbb{Z}_3[x]/(x^2+1)$. Suppose that its inverse is $[ax+b]$. Then $[x][ax+b] = [bx-a]$ (using the formula from Exercise 5). Thus, we need $a = -1$ and $b = 0$, so $[x]^{-1} = [-x]$.