MATH 456-01

Solutions to Homework 15

Section 6.1
p. 148: 1, 3, 11, 13, 17, 18, 19, 20, 29, 37

1. If $p(x)$ and $q(x)$ are constant polynomials, so are $p(x) - q(x)$ and $p(x)q(x)$; thus, K is a subring. However, since $x \cdot 1 = x \notin K$ even though $1 \in K$, K is not an ideal.

3. (a) Let $(k,0),(l,0) \in I$. Then $(k,0)-(l,0) = (k-l,0) \in I$ and $(k,0)(l,0) = (kl,0) \in I$. Therefore, I is an ideal of $\mathbb{Z} \times \mathbb{Z}$.

   (b) Since $(1,1) \in T$ but $(1,2)(1,1) = (1,2) \notin T$, T is not an ideal.

11. (a) $(0) = \{0\}, (1) = \mathbb{Z}_5 = (2) = (3) = (4)$. (This illustrates a general principle: if F is a field, then the only ideals of F are $\{0\}$ and F.)

    (b) $(0) = \{0\}, (1) = \mathbb{Z}_9 = (2) = (4) = (5) = (7) = (8), (3) = (6) = \{0,3,6\}$.

    (c) $(0) = \{0\}, (1) = \mathbb{Z}_{12} = (5) = (7) = (11), (2) = (10) = \{0,2,4,6,8,10\}, (3) = (9) = \{0,3,6,9\}, (4) = (8) = \{0,4,8\}, (6) = \{0,6\}$. Notice that $(m) = (\gcd(m,12))$.

13. Notice that if $m \in (n)$, in $\mathbb{Z}$, then $(m) \subseteq (n)$ by closure. For each $m \in \mathbb{Z}$, $(m) = (-m)$. Thus $(1) = \mathbb{Z} = (-1)$ even though $1 \neq -1$.

17. (a) If $a,b \in I \cap J$, then $a,b \in I$ and $a,b \in J$. Thus $a-b \in I$ and $a-b \in J$ since I and J are ideals, so $a-b \in I \cap J$. If $r \in R$, then $ra \in I$ and $rb \in J$ since I and J are ideals, so $ra \in I \cap J$. Therefore, $I \cap J$ is an ideal.

    (b) Let $a,b \in \bigcap_k I_k$, and let $r \in R$. Then $a,b \in I_k$ for each k, so $a-b \in I_k$ and $ra \in I_k$ for each k since I_k is an ideal. Therefore, $a-b, ra \in \bigcap_k I_k$, so $\bigcap_k I_k$ is an ideal.

18. $(2) \cup (3)$ is not a subring since $2,3 \in (2) \cup (3)$, but $2+3=5$ is in neither (2) nor (3).

19. Let $a,b \in I \cap S$ and let $r \in S$. Then $a,b \in I$ and $a,b \in S$, so $a-b \in I$ and $a-b \in S$. Thus $a-b \in I \cap S$. Also, since I is an ideal, $ra \in I$. Since S is a subring and $r,a \in S$, $ra \in S$, too. Therefore, $ra \in I \cap S$, so I is an ideal of S.

20. Since $0 \in I$ and $0 \in J$, $I, J \subseteq I+J$. Now let $x,y \in I+J$, $r \in R$. Then $x = a+b$ and $y = c+d$ for some $a,c \in I$ and $b,d \in J$. Thus $x-y = (a-c)+(b-d) \in I+J$ since $a-c \in I$ and $b-d \in J$ by closure of I and J under subtraction. Also, $rx = r(a+b) = ra+rb \in I+J$ since $ra \in I$ and $rb \in J$ (because I and J are ideals).

29. We know from exercise 15 that $(m) \cap (n)$ is an ideal. Suppose that $d \in (m) \cap (n)$. Then $m|d$ and $n|d$. Since m and n are relatively prime, this implies that $mn|d$. (See exercise 17 of Section 1.2.) Thus $d \in (mn)$, so $(m) \cap (n) \subseteq (mn)$. Conversely, if $d \in (mn)$, then $mn|d$, so $m|d$ and $n|d$. Therefore, $d \in (m) \cap (n)$. This gives us $(m) \cap (n) = (mn)$ if $(m,n) = 1$.

37. This is the converse of my remark in 7(a). Let $a \in R, a \neq 0_R$. Then $a \in (a)$, so $(a) \neq (0_R)$. Thus, by assumption, $(a) = R$. Since R has a unity 1_R, $1_R \in (a)$. That is, there exists $b \in R$ such that $ab = 1_R$. Therefore, a has a multiplicative inverse in R. Since we already know that R is a commutative ring with identity, R is in fact a field.