MATH 456-01

Solutions to Homework 16

Section 6.2
p. 159: 2, 5, 7, 12, 21, 26-29, (30-32).

2. Let $\phi : F \to R$ be a homomorphism, and let S be the image of ϕ. Let K be the kernel of ϕ. If $S = \{0_R\}$, then we are done. Assume that $S \neq \{0_R\}$. Since $\ker\phi$ is an ideal and F is a field, the kernel is either $\{0_F\}$ or F itself. However, if the kernel is F itself, then every element of F is mapped to 0_R; i.e., the image of ϕ would be $\{0_R\}$. Since this is not the case, the kernel must be $\{0_F\}$, so ϕ is one-to-one. Now $F/\ker\phi \cong S$ by the first isomorphism theorem. But since $F \cong F/\{0_F\}$, we have $F \cong S$, as desired.

5. This need not be true. For example, $\mathbb{Z}$ is an integral domain and (6) is an ideal of $\mathbb{Z}$, but $\mathbb{Z}/(6) \cong \mathbb{Z}_6$ is not an integral domain.

7. Define $\phi : R \to R$ by $\phi(r) = r$ for all $r \in \mathbb{R}$. ϕ is clearly surjective and $\ker\phi = (0_R)$, so by the first isomorphism theorem, $R/(0_R) \cong R$.

12. Let $x + I, y + I \in R/I$. Then $xy - yx \in I$, so $xy - yx = i$ for some $i \in I$. Thus $xy = yx + i$, so $xy \in yx + I$. We also know that $xy \in xy + I$. Since the cosets $xy + I$ and $yx + I$ are not disjoint, they must be equal. Now $(x + I)(y + I) = (xy) + I = (yx) + I = (y + I)(x + I)$, so R/I is commutative.

21. Define $\phi : \mathbb{Z}_{20} \to \mathbb{Z}_5$ by $\phi([n]_{20}) = [n]_5$. We must show first that ϕ is well-defined: if $[m]_{20} = [n]_{20}$, then $20|m - n$, so $5|m - n$ as well. Thus $[m]_5 = [n]_5$, and $\phi([m]_{20}) = \phi([n]_{20})$. ϕ is also a surjective homomorphism: if $[n]_5 \in \mathbb{Z}_5$, then $\phi([n]_{20}) = [n]_5$, and $\phi([m]_{20} + [n]_{20}) = \phi([m + n]_{20}) = [m + n]_5 = [m]_5 + [n]_5 = \phi([m]_{20}) + \phi([n]_{20})$, and similarly for multiplication.

    Finally, if $\phi([n]_{20}) = [0]_5$, then $[n]_5 = [0]_5$, so $[n]_{20} = [5k]_{20}$ for some $k \in \mathbb{Z}$. Thus $\ker\phi \subseteq ([5]_{20})$. Conversely, if $[n]_{20} \in ([5]_{20})$, then $[n]_{20} = [5k]_{20}$, so $\phi([n]_{20}) = [5k]_5 = [0]_5$. Therefore, $\ker\phi = ([5]_{20})$. By the First Isomorphism Theorem, $\mathbb{Z}_{20}/([5]_{20}) \cong \mathbb{Z}_5$.

    For Exercises 28-31, I will find a homomorphism between the given rings whose kernel is I.

26. Define $\phi : S \to \mathbb{Z}_2$ by $\phi(m/n) = [m]_2$, where m/n is in lowest terms. Then ϕ is well-defined since only one representative for each fraction is in lowest terms. (That is, if $a/b = c/d$ with a/b and c/d in lowest terms, then $a = c$ and $b = d$.) Clearly ϕ is surjective. Let $a/b, c/d \in S$, and let $q = \gcd(b, d)$. Then $\phi(a/b + c/d) = \phi\left(\dfrac{ad' + b'c}{b'd'q}\right) = [ad' + b'c]_2 = [a][d'] + [b'][c] = [a] + [c] = \phi(a/b) + \phi(c/d)$. Recall that b and d are odd by the definition of S, so $[b] = [b'] = [d] = [d'] = [1]$. (Before we could apply ϕ, we needed to get the gcd out of there so that our fraction was in lowest terms). We also get $\phi\left(\dfrac{a}{b} \cdot \dfrac{c}{d}\right) = \phi\left(\dfrac{ac}{bd}\right) = [ac] = [a][c] = \phi(a/b)\phi(c/d)$. Thus, ϕ is a homomorphism.

    Finally, $\ker\phi = I$ ($\phi(m/n) = [0]_2 \iff m$ is even), so $S/I \cong \mathbb{Z}_2$ by the First Isomorphism Theorem.

27. This is a generalization of Exercise 28. Please excuse any cut-and-paste errors.

    Define $\phi : T \to \mathbb{Z}_p$ by $\phi(m/n) = [m \cdot n^{-1}]_p$, where m/n is in lowest terms and n^{-1} is the multiplicative inverse of n mod p, which exists since $(p, n) = 1$. Then ϕ is well-defined since only one representative for each fraction is in lowest terms. (That is, if $a/b = c/d$ with a/b and c/d in lowest terms, then $a = c$ and $b = d$.) Clearly ϕ is surjective. Let $a/b, c/d \in T$, and let $q = \gcd(b, d)$. Then $b = b'q$ and $d = d'q$ for some $b', d' \in \mathbb{Z}$, so $\phi(a/b + c/d) = \phi\left(\dfrac{ad' + b'c}{b'd'q}\right) = [(ad' + b'c)(b'd'q)^{-1}]_p = [a(b'q)^{-1}] + [c(d'q)^{-1}] = [ab^{-1}] + [cd^{-1}] = \phi(a/b) + \phi(c/d)$. We also get $\phi\left(\dfrac{a}{b} \cdot \dfrac{c}{d}\right) = \phi\left(\dfrac{ac}{bd}\right) = [ac(bd)^{-1}] = [ab^{-1}][cd^{-1}] = \phi(a/b)\phi(c/d)$. Thus, ϕ is a homomorphism.

    Finally, $\ker\phi = I$ ($\phi(m/n) = [0]_p \iff p|m$), so $T/I \cong \mathbb{Z}_p$ by the First Isomorphism Theorem.

28. Define $\phi : T \to \mathbb{R}$ by $\phi\left(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\right) = a$. ϕ is clearly surjective. (If $a \in \mathbb{R}$, then $\phi\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) = a$.) ϕ is also well-defined since each such matrix has a unique representation.

Now $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} ac & ad+bc \\ 0 & ac \end{bmatrix}$, so

$$\phi\left(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\right)\phi\left(\begin{bmatrix} c & d \\ 0 & c \end{bmatrix}\right) = ac = \phi\left(\begin{bmatrix} ac & ad+bc \\ 0 & ac \end{bmatrix}\right)$$

and

$$\phi\left(\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\right) + \phi\left(\begin{bmatrix} c & d \\ 0 & c \end{bmatrix}\right) = a + c = \phi\left(\begin{bmatrix} a+c & b+d \\ 0 & a+c \end{bmatrix}\right).$$

Thus ϕ is a homomorphism. Now $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in \ker\phi$ if and only if $a = 0$, so the kernel is I. Thus, by the First Isomorphism Theorem, $T/I \cong \mathbb{R}$.

29. This is a generalization of Exercise 30. Please excuse any cut-and-paste errors.

Define $\phi : T \to \mathbb{R} \times \mathbb{R}$ by $\phi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = (a, c)$. ϕ is clearly surjective. (If $(x, y) \in \mathbb{R}$, then $\phi\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = (x, y)$.) ϕ is also well-defined since each such matrix has a unique representation.

Now $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} ac & ae+bf \\ 0 & cf \end{bmatrix}$, so

$$\phi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)\phi\left(\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right) = (ad, cf) = \phi\left(\begin{bmatrix} ac & ae+bf \\ 0 & cf \end{bmatrix}\right)$$

and

$$\phi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) + \phi\left(\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right) = (a, c) + (d, f) = (a + d, c + f) = \phi\left(\begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix}\right).$$

Thus ϕ is a homomorphism. Now $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \ker\phi$ if and only if $a = 0 = c$, so the kernel is I. Thus, by the First Isomorphism Theorem, $S/I \cong \mathbb{R} \times \mathbb{R}$.