MATH 456-01

Solutions to Homework 7.1

Section 7.1
p. 180: 1, 3, 4, 6, 8, 13, 16, 23, 29, 30

1. $e^{-1} = e, (12)^{-1} = (12), (13)^{-1} = (13), (23)^{-1} = (23), (123)^{-1} = (132), (132)^{-1} = (123)$ (as we saw in class).

3. (a) $|\mathbb{Z}_{18}| = 18$

   (b) A square has 8 symmetries, so $|D_4| = 2 \cdot 4 = 8$. Specifically, we can rotate by $0, 90, 180$, or 270 degrees, we can reflect across either diagonal, and we can reflect across a horizontal or vertical line bisecting the square.

   (c) $|S_4| = 4! = 24$.

   (d) $|S_5| = 5! = 120$.

   (e) $|U_{18}|\phi(18) = 6$. (There are six integers between 1 and 17, inclusive, that are relatively prime to 18. (1, 5, 7, 11, 13, 17.)

4. (a) Certainly the operation is associative, and G is closed under multiplication. Also, 6 is an identity since $2(6) = 2, 4(6) = 4, 6(6) = 6$, and $8(6) = 8$ in $\mathbb{Z}_{10}$. Also, each element has an inverse: $2^{-1} = 8$ and $4^{-1} = 4$. Therefore, G is a group.

   (b) Notice that $2^x(2^y 2^z) = 2^x 2^{y+z} = 2^{x+(y+z)} = 2^{(x+y)+z} = 2^{x+y}2^z = (2^x 2^y)2^z$, so the operation is associative. Since $0 \in \mathbb{Q}$, $2^0 = 1 \in G$, so we do have an identity (since the operation is multiplication). Also, If $2^x \in G$, then $x \in \mathbb{Q}$, so $-x \in \mathbb{Q}$. Therefore, $2^{-x} = (2^x)^{-1} \in G$. Therefore, G is a group.

   (c) The operation is ordinary multiplication, so it is associative. Since $1 \in G$ and the operation is multiplication, this set has an identity. However, $6 \in G$, but there is no $a \in G$ such that $6a = 1$. Therefore, G is not a group.

   (d) Let $a, b, c \in G$. Then $a*(b*c) = a*(b+c-bc) = a+(b+c-bc)-a(b+c-bc) = a+b+c-ab-ac-bc+abc$. On the other hand, $(a*b)*c = (a+b-ab)+c-(a+b-ab)c = a+b+c-ab-ac-bc+abc$. Therefore, the operation is associative. Notice that $a * 0 = a + 0 - a(0) = a$ for all $a \in G$, so 0 is an identity for this operation. Finally, given $a \in G$, we wish to find $b \in G$ such that $a * b = 0$. We solve:

$$\begin{aligned} a * b &= 0 \\ a + b - ab &= 0 \\ b(1-a) &= -a \\ b &= \frac{a}{a-1}. \end{aligned}$$

   Since $1 \notin G$, $\frac{a}{a-1} \in \mathbb{Q}$. Since $\frac{a}{a-1} \neq 1$ for any a, $\frac{a}{a-1} \in G$, and $\frac{a}{a-1} = a^{-1}$. Therefore, G is a group.

   (e) This operation is also associative; the proof is isomorphic to the proof for (d), and 0 is again an identity. Given $a \in G$, we seek $b \in G$ such that $a * b = 0$. Thus $a + b + ab = 0$, so $b = -\frac{a}{a+1} \in G$. Thus G is a group under this operation.

   (f) The operation is ordinary multiplication of complex numbers, so $*$ is associative. This also means that $1 \in G$ is the identity. ($1 \in G$ since $1(0) = 0$ and $1 + 0 \neq 0$.) The inverse of $c + di$ is $\frac{1}{c}$ or $\frac{-1}{d}i$ depending on whether $c \neq 0$ or $d \neq 0$. (The given condition on G indicates that exactly one of c and d is zero.)

8. We need elements relatively prime to the modulus in each case $U_4 = \{1, 3\}, U_6 = \{1, 5\}, U_{10} = \{1, 3, 7, 9\}, U_{20} = \{1, 3, 7, 9, 11, 13, 17, 19\}$, and $U_{30} = \{1, 7, 11, 13, 17, 19, 23, 29\}$.

13. $|S_3 \times \mathbb{Z}_2| = 12, |D_4 \times \mathbb{Z}_2| = 16, |S_3 \times \mathbb{Z}_5| = 30$, and $|S_4 \times \mathbb{Z}_2| = 48$.

16. Matrix multiplication is associative and we have the identity **1**. Thus, what we seek in the table is just confirmation that **1** appears in each row and column (so that every element has an inverse). In the table, I am leaving out the boldface.

$\cdot$	1	i	-1	$-i$	j	k	$-j$	$-k$
1	1	i	-1	$-i$	j	k	$-j$	$-k$
i	i	-1	$-i$	1	k	$-j$	$-k$	j
-1	-1	$-i$	1	i	$-j$	$-k$	j	k
$-i$	$-i$	1	i	-1	$-k$	j	k	$-j$
j	j	$-k$	$-j$	k	-1	i	1	$-i$
k	k	j	$-k$	$-j$	$-i$	-1	i	1
$-j$	$-j$	k	j	$-k$	1	$-i$	-1	i
$-k$	$-k$	$-j$	k	j	i	1	$-i$	-1

Since 1 does appear in each row and column, Q is a group.

23. Matrix multiplication is associative and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in SL(2,\mathbb{R})$. Also, since $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and $ad - bc = 1$, the inverse of a matrix in $SL(2,\mathbb{R})$ is also in $SL(2,\mathbb{R})$. Therefore, $SL(2,\mathbb{R})$ is a group.

29. In light of Exercise 26, you can approach this like a Sudoku puzzle.

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

30. The entries were determined in the order blue, red, green, purple, cyan, magenta. The purple entry comes from the fact that since e is the identity, if we find $xy = e$, we also get $yx = e$.

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	d	f	c
b	b	e				d
c	c	f	d			a
d	d	c				
f	f	d				

At this stage, we need to stop and think about groups. For example, $b^2 = b(b) = b(aa) = (ba)a = ea = a$, so $b^2 = a$. This gives us another entry. Below, I have changed all of the colored entries from above to blue. $b^2 = a$ is in green, and this allows us to fill in some more of the table (red).

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	d	f	c
b	b	e	a	f	c	d
c	c	f	d			a
d	d	c	f			
f	f	d	c			

Notice now that $f^2 = ff = (ca)f = c(af) = cc = c^2 = c(bd) = (cb)d = dd = d^2$. That is, $c^2 = d^2 = f^2$. Thus, all three remaining diagonal entries must equal b or e. However, if $c^2 = b$, then $cd = e$, which forces $dc = e$ and $f^2 = e \neq b$. Thus $c^2 = d^2 = f^2 = e$ (red). This forces the green below.

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	d	f	c
b	b	e	a	f	c	d
c	c	f	d	e	b	a
d	d	c	f	a	e	b
f	f	d	c	b	a	e