MATH 456-01
Solutions to Homework 22

Section 7.2

p. 201: 1, 2, 3, 6, 7, 9, 15, 17, 19, 30, 32, 35

1. Since every group element has an inverse, c has an inverse d. Thus $c^2 = c$ implies $d(c^2) = dc$, which in turn implies $(dc)c = dc$. Thus $ec = e$, so $c = e$.

2. $a = (132), b = (23)$. Thus $ab = (13)$, so $(ab)^{-1} = ab$. On the other hand, $a^{-1}b^{-1} = (123)(23) = (12) \neq (ab)^{-1}$!

3. $(abcd)^{-1} = d^{-1}c^{-1}b^{-1}a^{-1}$.

6. In S_3, $(12)^2 = (13)^2 = (23)^2 = e^2 = e$. We have **four** solutions!

7. (a) $5^1 = 5, 5^2 = 1$, so $o(5) = 2$.
   (b) $(1237645)^1 = (1237645), (1237645)^2 = (1365274), (1237645)^3 = (1753426), (1237645)^4 = (1624357), (1237645)^5 = (1472563), (1237645)^6 = (1546732), (1237645)^7 = e$. Therefore, $o((1237645)) = 7$. (In general, an n-cycle will have order n.) (Also, note that $(a_1a_2a_3\cdots a_n)^k = (a_1a_{k+1}a_{2k+1}\cdots a_{nk+1})$, where the subscripts are considered modulo n, provided $(n, k) = 1$. In general, you can compute the kth power of a cycle by just counting forward k steps from each element to see where that element is mapped.)
   (c) $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^4 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^5 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus $o\left(\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\right) = 6$.
   (d) $\begin{bmatrix} -1/2 & 1/2 \\ -3/2 & -1/2 \end{bmatrix}^2 = \begin{bmatrix} -1/2 & -1/2 \\ 3/2 & -1/2 \end{bmatrix}, \begin{bmatrix} -1/2 & 1/2 \\ -3/2 & -1/2 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $o\left(\begin{bmatrix} -1/2 & 1/2 \\ -3/2 & -1/2 \end{bmatrix}\right) = 3$.

9. (a) $|U_{10}| = 4(1, 3, 7, 9), |U_{12}| = 4, (1, 5, 7, 11), |U_{24}| = 8(1, 5, 7, 11, 13, 17, 19, 23)$.
   (b) $o(1) = 1, o(3) = 4, o(7) = 4, o(9) = 2, o(11) = 2, o(13) = 4, o(17) = 4, o(19) = 2$.

15. (a) $o(a)$ could be any divisor of 12: 1, 2, 3, 4, 6, or 12.
    (b) $o(b) = p$ since $o(b)|p$ but $o(b) \neq 1$.

17. (a) $x = a^{-1}b$ is a solution for $ax = b$ and $y = ba^{-1}$ is a solution for $ya = b$. If x' is another solution to $ax = b$, then $ax = ax' \implies x = x'$. Similarly, if y' is a second solution for $ya = b$, then $ya = y'a \implies y = y'$. Thus, in both cases, the solution is unique.
    (b) If $a = (132)$ and $b = (13)$, $x = (23)$ is the solution to $ax = b$ from Exercise 2. To get $ya = b$, we need $y = ba^{-1} = (13)(123) = (12) \neq x$.

19. Note that $(bab^{-1})^n = (bab^{-1})(bab^{-1})\cdots(bab^{-1}) = ba^nb^{-1}$. If $o(a) = n$, then $(bab^{-1})^n = ba^nb^{-1} = beb^{-1} = e$, so $o(bab^{-1}) \leq n$. On the other hand, if $m = o(bab^{-1})$, then $(bab^{-1})^m = ba^mb^{-1} = e$, so $ba^m = b$ and $a^m = e$. Thus $m \geq n$. Therefore, $o(bab^{-1}) = o(a)$.

30. Let $n = o(ab), m = o(ba)$. Then $(ba)^{n+1} = (ba)(ba)\cdots(ba) = b(ab)(ab)\cdots(ab)a = b(ab)^na = bea = ba$. That is, $(ba)^{n+1} = ba$, so $(ba)^n = e$. Therefore, $m \leq n$. Reversing the roles of a and b shows that $n \leq m$, as well, so $m = n$.

32. Note first that the elements of order two are precisely the non-identity elements that are their own inverse. In any group, every element has an inverse. Let n be the number of non-identity elements that are self-inverse, and let $b_1, \ldots, b_k$ have inverses $c_1, \ldots, c_k$, respectively, where $\{b_1, \ldots, b_k\}$ and $\{c_1, \ldots, c_k\}$ are disjoint sets. Then $|G| = 1 + n + 2k$. (The 1 is for the identity element.) Since $|G|$ is even, $n \geq 1$, so some element has order two.

35. If we can show that $b^3 = e$, it will follow immediately that $ab = ba$ since we already know that $ab = b^4a$. Now $ab = b^4a \implies b^2ab = b^6a = a$. Now $(b^2ab)b^5 = ab^5$, so $b^2a = ab^5 = ab(b^4) = (b^4a)b^4 = b^4(ab)b^3 = b^4(b^4a)b^3 = b^8ab^3 = b^2ab^3$. Thus $e = b^3$.