MATH 456-01
Solutions to Homework 23

Section 7.3
p. 211: 11, 13, 15, 16, 17, 19, 22, 27, 33, 50

11. (a) $(1,1) = (1,1), 2(1,1) = (2,2) = (0,2), 3(1,1) = (3,3) = (1,0), 4(1,1) = (0,1), 5(1,1) = (1,2), 6(1,1) = (0,0)$. Thus $(1,1)$ generates all of $\mathbb{Z}_2 \times \mathbb{Z}_3$, so $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic.

    (b) $\mathbb{Z}_2 \times \mathbb{Z}_4$ is generated by $(1,0)$ and $(0,1)$. However, it is not cyclic: if $(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_4$, then $4(a,b) = (0,0)$, but $|\mathbb{Z}_2 \times \mathbb{Z}_4| = 8$.

13. Certainly $e_H \in G$, and $e_H e_H = e_H$ since e_H is the identity of H. By Exercise 1 of 7.2, $e_H = e_G$.

15. I will apply the one-step subgroup test in each case.

    (a) Since the identity is in every subgroup of G, $H \cap K$ is nonempty. Let $a, b \in H \cap K$. Then $a, b \in H$ and $a, b \in K$, and $ab^{-1} \in H$ and $ab^{-1} \in K$ since both are subgroups of G. Thus $ab^{-1} \in H \cap K$, so $H \cap K \leq G$.

    (b) As before, $\cap H_i$ is nonempty since it contains the group identity. If $a, b \in \cap H_i$, then $a, b \in H_i$ for each i. Thus $ab^{-1} \in H_i$ for each i since H_i is a subgroup. Therefore, $ab^{-1} \in \cap H_i$, and $\cap H_i \leq G$.

16. Since $G_1 \leq G$ and $H_1 \leq H$, G_1 and H_1 are nonempty, so $G_1 \times H_1$ is also nonempty. Let $a, b \in G_1 \times H_1$. Then $a = (g_1, h_1), b = (g_2, h_2)$ for some $g_1, g_2 \in G_1$ and $h_1, h_2 \in H_1$. Now $ab^{-1} = (g_1, h_1)(g_2^{-1}, h_2^{-1}) = (g_1 g_2^{-1}, h_1 h_2^{-1}) \in G_1 \times H_1$ since G_1 and H_1 are groups.

17. Suppose that $n \in \mathbb{Z}$ is a generator of $\mathbb{Z}$. Then there is an integer m such that $mn = 1$. Thus $m, n \in \{\pm 1\}$. In particular, $n = \pm 1$.

19. $T \neq \emptyset$ since $e \in T$. Let $a, b \in T$, and suppose that $o(a) = m, o(b) = n$. Then $(ab^{-1})^{mn} = a^{mn}(b^{-1})^{mn}$ (using the fact that G is abelian), which becomes $(a^m)^n((b^n)^m)^{-1} = ee^{-1} = e$. Therefore, $T \leq G$.

22. Let $b \in G$. We must show that $ab = ba$. Consider $b^{-1}ab$. Squaring this gives $(b^{-1}ab)^2]b^{-1}a^2b = b^{-1}eb = e$. Thus $b^{-1}ab = e$ or $b^{-1}ab = a$ since a is the only element of order 2. In the first case, we get $ab = be = b$, so $a = e$, which is false. Thus $b^{-1}ab = a$, so $ab = ba$.

27. Since $H \neq \emptyset, x^{-1}Hx \neq \emptyset$. Let $a, b \in x^{-1}Hx$. Then $a = x^{-1}hx, b = x^{-1}kx$ for some $h, k \in H$. Now $ab^{-1} = (x^{-1}hx)(x^{-1}kx)^{-1} = (x^{-1}hx)(x^{-1}k^{-1}x) = x^{-1}hk^{-1}x \in x^{-1}Hx$, so $x^{-1}Hx \leq G$.

33. Since $e \in C(a), C(a) \neq \emptyset$. I will apply the two-step subgroup test. Let $x, y \in C(a)$. Then $(xy)a = x(ya) = x(ay) = (xa)y = (ax)y = a(xy)$, so $xy \in C(a)$. Since $ax = xa$, $x^{-1}a = ax^{-1}$, so $x^{-1} \in C(a)$, as well. Therefore, $C(a) \leq G$.

50. Suppose that $\frac{a}{b} \in \mathbb{Q}$ generates $\mathbb{Q}$. Then there is a positive integer n such that $n\frac{a}{b} = \frac{a}{2b}$. But this implies that $2abn = ab$, so $2n = 1$, which is impossible. Therefore, $\mathbb{Q}$ is not cyclic.