

# MATH 456-01

## Solutions to Homework 24

### Section 7.4

p. 196: 1, 4, 5, 9, 15, 24, 25, 27, 28, 29

1. (a) If  $f(x) = f(y)$ , then  $3x = 3y \implies x = y$ , so  $f$  is injective. If  $y \in \mathbb{R}$ , then  $f(y/3) = 3(y/3) = y$ , so  $f$  is surjective.  $f(x+y) = 3(x+y) = 3x+3y = f(x)+f(y)$ .  
 (b) Here,  $f(xy) = 3xy$ , but  $f(x)f(y) = (3x)(3y) = 9xy$ .
13. Note that  $U_{10} = \langle 3 \rangle$  and  $|U_{10}| = 4$ , so  $U_{10} \cong \mathbb{Z}_4 \cong U_5$ .
19. Suppose now that  $G$  is abelian and  $x, y \in G$ . Then  $f(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = f(x)f(y)$ .  
 Conversely, Suppose that  $f(x) = x^{-1}$  defines a homomorphism. Let  $x, y \in G$ . Then  $x^{-1}, y^{-1} \in G$ , and  $xy = (y^{-1}x^{-1})^{-1} = f(y^{-1}x^{-1}) = f(y^{-1})f(x^{-1}) = yx$ . Therefore,  $G$  is abelian.  
 Since every element has a unique inverse,  $f(x) = x^{-1}$  defines a bijective function, so we have an isomorphism.
36. We already know that composition is associative. From Exercise 9, we know that a composition of isomorphisms is an isomorphism, so  $\text{Aut } G$  is closed under composition. The identity function is an isomorphism that acts as an identity for the group (as we saw in class). Finally, every isomorphism has an inverse that is also an isomorphism. In this case, that means that every automorphism has an inverse that is also an automorphism. Therefore,  $\text{Aut } G$  is a group under composition.
37. Since inner automorphisms exist,  $\text{Inn } G$  is nonempty. Suppose  $f, g \in \text{Inn } G$ . Then there are group elements  $c, d$  such that  $f$  and  $g$  are given by  $f(a) = c^{-1}ac, g(a) = d^{-1}ad$ . Now  $f \circ g^{-1}(a) = f(dad^{-1}) = c^{-1}(dad^{-1})c = (c^{-1}d)a(d^{-1}c) = (d^{-1}c)^{-1}a(d^{-1}c)$ . Thus  $f \circ g^{-1} \in \text{Inn } G$ .
40.  $\mathbb{Z}_6$  is abelian and  $S_3$  is not.
41.  $\mathbb{Z}_4 \times \mathbb{Z}_2$  is abelian and  $D_4$  is not.
42.  $\mathbb{Z}_4 \times \mathbb{Z}_2$  has an element of order 4 (namely,  $(1, 0)$ ), while  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$  does not.
43.  $U_8 = \{1, 3, 5, 7\}, U_{10} = \{1, 3, 7, 9\}$ . In  $U_8$ , every element except 1 has order 2. In  $U_{10}$ ,  $o(3) = 4$ .
44.  $U_{12} = \{1, 5, 7, 11\}$ . Every element of  $U_{12}$  except 1 has order 2, so  $U_{10}$  is not isomorphic to  $U_{12}$ .
45.  $U_8 \cong U_{12}$ . The operation tables are shown below.

|   | $\mathbb{Z}_8$ |   |   |   |    | $\mathbb{Z}_{12}$ |    |    |    |
|---|----------------|---|---|---|----|-------------------|----|----|----|
|   | 1              | 3 | 5 | 7 |    | 1                 | 5  | 7  | 11 |
| 1 | 1              | 3 | 5 | 7 | 1  | 1                 | 5  | 7  | 11 |
| 3 | 3              | 1 | 7 | 5 | 5  | 5                 | 1  | 11 | 7  |
| 5 | 5              | 7 | 1 | 3 | 7  | 7                 | 11 | 1  | 5  |
| 7 | 7              | 5 | 3 | 1 | 11 | 11                | 7  | 5  | 1  |

Thus  $1 \mapsto 1, 3 \mapsto 5, 5 \mapsto 7, 7 \mapsto 11$  gives an isomorphism.