MATH 456-01
Solutions to Homework 25

Section 8.1
p. 206: 2-5, 7-11, 17, 19, 23, 25, 35

2. I will use the operation table on page 190. Since $|D_4| = 8$ and $|K| = 2$, there are four cosets: $Kr_0 = \{r_0, v\}, Kr_1 = \{r_1, t\}, Kr_2 = \{r_2, h\}$, and $Kr_3 = \{r_3, d\}$.

3. This time there will be 2 cosets: $Kr_0 = \{r_0, r_1, r_2, r_3\}$ and $Kd = \{d, h, t, v\}$.

4. Note first that $K = \{e, (23)\}$. $|S_3| = 6$ and $|K| = 2$, so there are 3 cosets: $Ke = \{e, (23)\}, K(123) = \{(123), (13)\}$, and $K(132) = \{(132), (12)\}$.

5. $|U_{32}| = 16$ and $|K| = 2$, so there are 8 cosets. They are: $K1 = \{1, 17\}, K3 = \{3, 19\}, K5 = \{5, 21\}, K7 = \{7, 23\}, K9 = \{9, 25\}, \{11, 27\}, \{13, 29\}$, and $\{15, 31\}$.

7. $[G : H] = |D_4|/|H| = 8/2 = 4$.

8. $[G : H] = |Z_{12}|/|H| = 12/4 = 3$.

9. $[G : H] = |Z_{12}|/|H| = 12/12 = 1$. $(3, 20) = 1$.

10. $[G : H] = |Z_{40}|/|H| = 40/10 = 4$. $(12, 20) = 4$, so $< 4 > \leq H$. Conversely, $H \leq < 4 >$, so $H = < 4 >$.

11. $H = < (1234) >$. Since $o((1234)) = 4$, $|H| = 4$. Thus $[G : H] = |S_4|/|H| = 24/4 = 6$.

17. (a) The divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24, so these are the possible orders of subgroups of $\mathbb{Z}_{24}$.
    (b) Since $|S_4| = 24$, the possible subgroup orders are the same as in (a).
    (c) $|D_4 \times \mathbb{Z}_{10}| = 80$, so the possible subgroup orders are $1, 2, 4, 5, 8, 10, 16, 20, 40, 80$.

19. The least common multiple of $1, 2, 3, \ldots, 12$ is $8(9)(5)(7)(11) = 27720$.

23. Let $a \in G, a \neq e$. Then $< a > \leq G$, so $< a > = \{e\}$ or $< a > = G$. Since $a \neq e, < a > = G$. Therefore, G is cyclic, so $G \cong \mathbb{Z}_n$ for some integer n or $G \cong \mathbb{Z}$. But $\mathbb{Z}$ has proper subgroups, so $G \cong \mathbb{Z}_n$ for some n. If n is not prime, then $\mathbb{Z}_n$ has a proper subgroup, so n must be prime.

25. $| < a^4 > | = o(a^4) = o(a^{(4,30)}) = o(a^2) = 15$. Thus $[< a > : < a^4 >] = | < a > |/| < a^4 > | = 30/15 = 2 = (4, 30)$.

35. Let G be a group of order pq, with p and q distinct primes, and let $H \leq G$. Since $|H|$ divides $|G|$, $|H| = p$ or $|H| = q$. Thus H can have no proper subgroups by Lagrange's Theorem, so H is cyclic by Exercise 11.