

MATH 456-01
Solutions to Homework 26

Section 8.2

p. 213: 7, 8, 21, 26, 30

7. We must show that $G^* \leq G \times H$ and that it is normal in $G \times H$. Certainly G^* is nonempty, and if $x = (a, e), y = (b, e) \in G^*$, then $xy^{-1} = (a, e)(b^{-1}, e^{-1}) = (ab^{-1}, e) \in G^*$ since $ab^{-1} \in G$. Thus $G^* \leq G \times H$. Let $(g, h) \in G \times H$, and let $(a, e) \in G^*$. Then $(g, h)^{-1}(a, e)(g, h) = (g^{-1}ag, h^{-1}eh) = (g^{-1}ag, e) \in G^*$. Therefore, G^* is a normal subgroup of $G \times H$.
8. (a) The cyclic subgroups of Q are $\langle 1 \rangle = \{1\}, \langle -1 \rangle = \{1, -1\}, \langle i \rangle = \{i, -1, -i, 1\} = \langle -i \rangle, \langle j \rangle = \{j, -1, -j, 1\} = \langle -j \rangle, \langle k \rangle = \{k, -1, -k, 1\} = \langle -k \rangle$.
- (b) If $a \in Q$, then $a^{-1} \langle 1 \rangle a = \{a^{-1}1a\} = \{1\} = \langle 1 \rangle$. Also, $a^{-1} \langle -1 \rangle a = \{a^{-1}1a, a^{-1}(-1)a\} = \{1, -1\} = \langle -1 \rangle$. The groups $\langle i \rangle, \langle j \rangle, \langle k \rangle$ all behave similarly to each other, so I will just show that $\langle i \rangle \trianglelefteq Q$. We have $a^{-1} \langle i \rangle a = \{a^{-1}ia, a^{-1}(-1)a, a^{-1}(-i)a, a^{-1}1a\} = \{\pm i, -1, -mpi, 1\} = \langle i \rangle$. (I used the table from 7.1 Exercise 14 to do my computations.) Alternatively, $\langle i \rangle, \langle j \rangle, \langle k \rangle$ all have order 4 and hence index 2. By exercise 20, that makes them all normal.
21. Let $n \in N, k \in K$. Then $(nkn^{-1})k^{-1} \in K$ since K is normal, and $n(kn^{-1}k^{-1}) \in N$ since N is normal. Thus $nkn^{-1}k^{-1} \in K \cap N = \{e\}$, so $nkn^{-1}k^{-1} = e$. Therefore, $nk = kn$ for all $n \in N, k \in K$.
26. Note that $a^{-1}Na \leq G$: it is nonempty, and if $x, y \in a^{-1}Na$, then $x = a^{-1}n_1a, y = a^{-1}n_2a$ for some $n_1, n_2 \in N$. Thus $xy^{-1} = (a^{-1}n_1a)(a^{-1}n_2a)^{-1} = (a^{-1}n_1a)(a^{-1}n_2^{-1}a) = a^{-1}(n_1n_2^{-1})a \in a^{-1}Na$. In addition, conjugation induces a bijection on G : if $a^{-1}xa = a^{-1}ya$, then $x = y$, and if $y \in G$, then $a^{-1}(aya^{-1})a = y$. Therefore, $a^{-1}Na$ is a subgroup of order n since N was. But N is the **only** subgroup of G of order n , so $a^{-1}Na = N$. By Theorem 7.34, $N \trianglelefteq G$.
30. Following the hint, let us define $f : A \times B \rightarrow G$ by $f(a, b) = ab$. Since $AB = G$, f is surjective. If $f(a, b) = f(c, d)$, then $ab = cd$, so $c^{-1}a = db^{-1} \in A \cap B$. Thus $c^{-1}a = e_A$ and $db^{-1} = e_B$, so $(a, b) = (c, d)$, implying that f is injective.
- Now $f((a, b)(c, d)) = f(ac, bd) = (ac)(bd) = (ab)(cd) = f(a, b)f(c, d)$ (using Exercise 19). Therefore, f is an isomorphism of groups.