

MATH 456-01
Solutions to Homework 27

Section 8.3

p. 260: 4, 5, 6, 11, 23, 31

4. Since N has index 2, it is normal in S_3 . Its cosets are N and $N(12) = \{(12), (13), (23)\}$. Since $NN = N$, $NN(12) = N(12)$, $N(12)N = N(12)$, and $N(12)N(12) = N$, $S_3/N \cong \mathbb{Z}_2$.
5. $\langle 6 \rangle = \{0, 6, 12\}$. The cosets are $N, N+1, N+2, N+3, N+4$, and $N+5$, and the Cayley table looks the same as that of \mathbb{Z}_6 .
6. The cosets are $N, N+1$, and $N+2$. The Cayley table looks just like the table for \mathbb{Z}_3 .
11. If $a+N, b+N$ are cosets of N , then $(a+N) + (b+N) = (a+b) + N = (b+a) + N = (b+N) + (a+N)$ since G is abelian. Thus G/N is also abelian.
23. R^*/R^{**} is isomorphic to \mathbb{Z}_2 . The cosets are \mathbb{R}^* and $(-1)\mathbb{R}^*$. For if $aR^{**} = bR^{**}$, then $ab^{-1} \in \mathbb{R}^{**}$, which means that $ab^{-1} > 0$. Thus a and b^{-1} (and hence a and b) must have the same sign. Since only two signs are available, there are only two cosets. ($ab^{-1} \notin \mathbb{R}^{**} \implies ab < 0$, so a and b have opposite signs. I.e., $a \in \mathbb{R}^{**}$ if and only if $a > 0$.)
31. If $Z(G) = \{e\}$ or $Z(G) = G$, we are done, so we may assume $Z(G)$ is a proper subgroup of G . Since $Z(G) \trianglelefteq G$, $G/Z(G)$ is a group. Since the order of G is pq , $|Z(G)|$ is p or q . This implies that $|G/Z(G)|$ is q or p , respectively. However, p and q are primes, so $G/Z(G)$ must be cyclic by Exercise 11 of 7.5 (since a group of prime order can have no proper subgroups). Now by Theorem 7.38, G is abelian, so $Z(G) = G$. This is a contradiction since we assumed $Z(G)$ was a proper subgroup of G , so $Z(G) = G$ or $Z(G) = \{e\}$.