

**MATH 456-01**  
**Solutions to Homework 28**

**Section 8.4**

**p. 270: 1, 8, 9, 10, 14, 17, 28, 39**

1.  $f(a + bi) = b = 0$  if and only if  $b = 0$ , so  $\ker f = \mathbb{R}$ .  $f((a + bi) + (c + di)) = f((a + c) + (b + d)i) = b + d = f(a + bi) + f(c + di)$ .
8.  $f(x + y) = 3(x + y) = 3x + 3y = f(x) + f(y)$ . Certainly  $\langle 4 \rangle \subseteq \ker f$ . If  $f(x) = 0$ , then  $3x = 0$  in  $\mathbb{Z}_{12}$ , so  $12|3x$ . Thus  $4|x$ , so  $x \in \langle 4 \rangle$ . Thus  $\ker f = \langle 4 \rangle$ .
9.  $f(k + l) = ([k + l]_2, [k + l]_4) = ([k]_2, [k]_4) + ([l]_2, [l]_4) = f(k) + f(l)$ , so  $f$  is a homomorphism. If  $k \in \ker f$ , then  $k$  is congruent to 0 mod 2 and mod 4, so  $4|k$ . Conversely, if  $4|k$ , then  $k \in \ker f$ . Thus  $\ker f = \langle 4 \rangle$ .
10. We must check that  $\phi(fg) = \phi(f)\phi(g)$ . Notice that since  $fg \in S_n$ ,  $\phi(fg)(k) = fg(k)$  if  $1 \leq k \leq n$  and  $n + 1$  if  $k = n + 1$ . On the other hand,  $\phi(f)\phi(g)(k) = \phi(f)(g(k))$  if  $k < n + 1$  and  $\phi f(n + 1)$  if  $k = n + 1$ . In the first case, since  $g(k) < n + 1$ ,  $\phi(f)(g(k)) = f(g(k)) = fg(k)$ . In the second case,  $\phi f(n + 1) = n + 1$ . In both cases, we get the same result as we did for  $\phi(fg)$ , so  $\phi$  is a homomorphism. Its kernel is just  $\{e\}$  since no other permutation in  $S_n$  fixes every element.
14.  $h(x + y) = 2(x + y) = 2x + 2y = h(x) + h(y)$ , where the computations are being performed in the appropriate groups. Thus  $h$  is a homomorphism.  $x \in \ker h$  if and only if  $h(x) = [2x]_3 = [0]_3$ . Since  $3|2x$  if and only if  $3|x$ , we have  $\ker h = \langle [3]_{18} \rangle$ .
17. (a) The kernels of such homomorphisms have orders dividing 12, so the images must also have orders dividing 12. Since  $\mathbb{Z}_{12}$  is cyclic, so are all of its quotient groups and hence, by the First Isomorphism Theorem, its homomorphic images are, too. Thus the images must be cyclic groups of orders dividing 12. Since cyclic groups of a given order are unique up to isomorphism, our list is  $\{0\}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ , and  $\mathbb{Z}_{12}$ .  
(b) Reasoning as in (a), we see that the possible homomorphic images of  $\mathbb{Z}_{20}$  are  $\{0\}, \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_{10}$ , and  $\mathbb{Z}_{20}$ .
28. Define  $\phi : GL(2, \mathbb{R}) \rightarrow \mathbb{R}^*$  by  $\phi(M) = \det M$  for each  $M \in GL(2, \mathbb{R})$ .  $\phi$  is well-defined since members of  $GL(2, \mathbb{R})$  have nonzero determinant. It is also a homomorphism since  $\det(MN) = \det(M)\det(N)$ , and it is surjective since  $\det \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = a$ .  $M \in \ker \phi$  if and only if  $\det M = 1$ , which is precisely how  $SL(2, \mathbb{R})$  is defined. Thus, by the First Isomorphism Theorem,  $GL(2, \mathbb{R})/SL(2, \mathbb{R}) \cong \mathbb{R}^*$ .
39. Following the hint, we see that  $T$  is one-to-one since  $T(f(x)) = T(g(x)) \implies \mathbb{Z} + xf(x) = \mathbb{Z} + xg(x) <$  so  $x(g(x) - f(x)) \in \mathbb{Z}$ . This can only happen if  $g(x) = f(x)$ . It is also surjective: if  $f(x) + \mathbb{Z} \in \mathbb{Z}[x]/\mathbb{Z}$ , then  $f(x) = xg(x) + n$  for some  $g(x) \in \mathbb{Z}[x], n \in \mathbb{Z}$  by the division algorithm. Thus  $f(x) + \mathbb{Z} = xg(x) + \mathbb{Z}$ , and  $T(g(x)) = xg(x) + \mathbb{Z} = f(x) + \mathbb{Z}$ .  
Finally,  $T$  is a homomorphism since  $T(f(x) + g(x)) = \mathbb{Z} + (f(x) + g(x)) = (\mathbb{Z} + f(x)) + (\mathbb{Z} + g(x)) = T(f(x)) + T(g(x))$ . Therefore  $\mathbb{Z}[x] \cong \mathbb{Z}[x]/\mathbb{Z}$ .