The Radius Ratio Dependence of the Primary Instability in the Taylor Couette System

by

Andrew Lum

Submitted in Partial Fulfillment of the
Requirements for the Degree

Bachelor of Arts

Supervised by
Dr. Daniel Borrero-Echeverry

Department of Physics

Willamette University, College of Liberal Arts
Salem, Oregon

2017

Presentations and publications

- "The Radius Ratio Dependence of the Primary Instability in the Taylor-Couette System," Student Scholarship Recognition Day: 2017.
- "The Radius Ratio Dependence of the Primary Instability in the Taylor-Couette System," Physics Senior Theses Poster Presentation: 2017.

Acknowledgments

I would like to thank and acknowledge Professor Borrero-Echeverry for advising me on my thesis, and all the various other things he helped me with. Thanks to the Willamette University Physics Department for all their help with everything and providing a great place to learn. Finally I would also like to thank my family for the endless support they gave me.

Abstract

Technical: A Taylor-Couette system is a system that contains fluid in the gap between two co-axial rotating cylinders. A Taylor-Couette system undergoes a transition from a smooth laminar flow, known as circular Couette flow, into a vortex flow, known as Taylor vortex flow, when the rotation rates of the cylinders reach the critical Reynolds number, Re_{c}. The Reynolds number, Re, is a function of angular velocity, which describes when the centrifugal forces overcome the viscous damping and the system transitions. This transition is known as the primary instability, and is affected by the geometry of the Taylor-Couette system. Through the use of 3-D printed shells that are attached to the inner cylinder, this project examined how the primary instability threshold is affected by changing the radius ratio of the cylinders. We then measured the angular velocities, and calculated the Re_{c}, where the transition between circular Couette flow into Taylor vortex flow occurred. Our data suggest that increasing the radius ratio also increases the Re_{c}.

General: The Taylor-Couette system is made up of two rotating cylinders, an inner cylinder and an outer cylinder. In between the inner and outer cylinders is a gap filled with fluid. Depending on the rotations rates of the cylinders, the fluid in the gap exhibits specific flow states. At low rotations rates, the fluid state is a smooth laminar flow. As the rotation rates of the cylinders increase past a specific angular velocity, the fluid state transitions to a vortex flow. This transition is known as the primary instability of the system. The primary instability is influenced by the geometry of the system. Therefore, by adjusting the radii of the cylinders, we can change the angular velocity needed to cause the system transition from laminar to vortex flow. This project examined how the primary instability threshold is affected by adjusting the radii of the cylinders through the use of 3D printed shells that attach to the inner cylinder. We then measured the angular velocities needed to cause the system to undergo transition and found that if the radius ratio increases, so does the angular velocity needed to cause transition.

Table of Contents

List of Figures

1 Introduction

Turbulent flow refers to a flow regime that is random, nonlinear, diffusive, dissipative, and has high levels of fluctuating vorticity. Turbulence is present within a variety of natural occurring flows like ocean currents, and man-made flows, such as the wake left behind a boat. Turbulence occurs when the inertial forces within a flow become strong enough to overcome the effects of viscosity [CKA07]. The relative importance of inertial effects compared to viscous effects is captured by the dimensionless quantity known as the Reynolds number, Re. A small Re indicates that the viscous forces are stronger than the inertial forces, and results in a laminar flow. A laminar flow is characterized by layers of fluid that slide over one another resulting in an overall smooth flow. As Re becomes larger, the inertial forces overcome the viscous forces and the laminar flow changes into a turbulent flow [Eck07].

Knowing when a particular flow is going to become turbulent is useful for a variety of practical reasons. For example the prediction of when turbulence will occur is used in weather prediction. Deriving a general theory that is able to predict when a flow will become turbulent is not only important but it is still ongoing. The transition from laminar to turbulent flow was studied in the past[1], and is still is being studied. Research into this problem led scientists to look for easier to control, and mathematically tractable, systems to study turbulence with. The search for systems that followed these criteria eventually lead to the discovery of the Taylor Couette system [Tag94].

The Taylor Couette system is a widely studied flow system consisting of two co-axial independently rotating cylinders. The configuration of the cylinders are shown in Fig. 1.1. Where h is the height of the system, r_{i} is the radius of the inner cylinder, and r_{o} is the radius of the outer cylinder. In between the inner and outer cylinders, there is a gap where the fluid flow is located. This system is well known for its ability to produce a wide variety of flow states depending

[1] One example of this is the research of Bénard cells [Kos93] and [BE]. More recent research has focused on the subcritical transition to turbulence in shear flows. This is discussed in [Cha02] and [BC12]

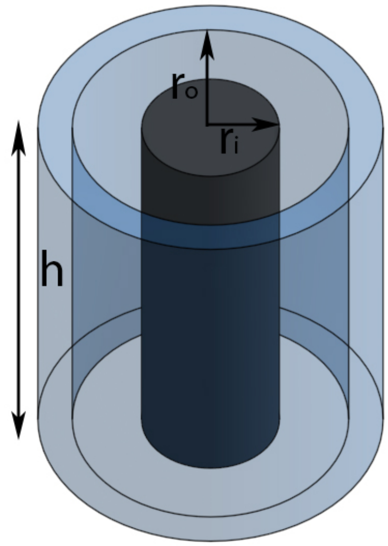

Figure 1.1: An example of a basic Taylor Couette system. There are two independently rotating cylinders, an outer one and an inner one. In between the two cylinders is a gap where fluid is able to flow. As for the variables, h is the height of the system, r_{i} is the radius of the inner cylinder, and r_{o} is the radius of the outer cylinder.

on the rotation rates and relative sizes of the cylinders [ALS86]. As one or more of the cylinders begins to rotate slowly, the resulting flow in between the two cylinders becomes a laminar flow known as circular Couette flow. If the cylinder rotation rates continue to increase, then the laminar flow changes to a vortex flow called Taylor vortex flow. The point where circular Couette flow changes into a Taylor vortex flow is known as the primary instability of the system. As the rotation rates of one or more of the cylinders increases, the vortex flow becomes increasingly unstable and more chaotic [Tag94].

Various factors can influence the onset of primary instability of the Taylor Couette system. These factors not only include the rotation rates of the cylinders, but also the geometry of the system characterized by Γ, the height of the test cell over the gap width, and η the radius ratio, given by radius of the inner cylinder divided by the radius of the outer cylinder [Tag94]. Most research; however, focuses on the parts of the cylinders that are easier to control, such as the rotation rates of the system. This leaves the effect the system's geometry has on the primary instability relatively untested.[2]

My thesis looks at the effect *eta* has on transition between circular Couette flow

[2]The radius ratio dependance was previously studied by Taylor who used wax to form the inner cylinder [Tay23]. This was also researched by [LTK+87] and [MPGL14] later.

and Taylor couette flow. The next chapter focuses on the background information related to my thesis.

2 Background

This section focuses on what Taylor Couette flow is, its related equations, and what the primary instability is and how η effects it.

2.1 Taylor Couette Flow

Taylor Couette flow refers to the fluid flow within the system that was originally designed in 1888 by A. Mallock [Mal88] and M. Couette [PBCP94]. Originally used for examining the viscous behavior of fluids, a Taylor Couette system consists of two independently rotating cylinders as shown in Fig. 1.1. The flow created between rotating cylinders is interesting because small increases to the inner cylinder's angular velocity can produce easily distinguishable fluid states of increasing complexity [ALS86]. These fluid states vary considerably. A low inner angular velocity results in a simple featureless flow. This flow is referred to as circular Couette flow and can be seen in Fig. 2.1. As the speed of the inner cylinder increases past a critical point, the circular Couette flow changes to a flow characterized by toroidal vortices [Kos93].

This change from an axially symmetric flow to a stack of toroidal vortices was studied by G.I. Taylor in 1923, where he was able to quantitatively predict and experimentally confirm the existence of a flow instability based on the speed at which the inner cylinder rotates. Taylor found that as the flow became unstable, it was replaced with a pattern where the fluid traveled in layered vortices following helical paths around the cylinder [Tay23]. The vortices are a result of inertial forces pushing fluid outwards until the fluid meets the outer cylinder. When this happens, the fluid is forced to overturn on itself, which creates the vortex pattern [Tag94]. These vortex patterns are now known as Taylor vortices, which can be seen in Fig. 2.2.

Initial experiments using the system were done by only rotating the inner cylinder. However, a wide variety of flow regimes can be found by simply varying the rotation rates of one or both of the cylinders [ALS86]. The rotation rates of the cylinders are not the only factors that can influence which flow regime is exhibited within the Taylor Couette system, the geometry of the system can also play a role.

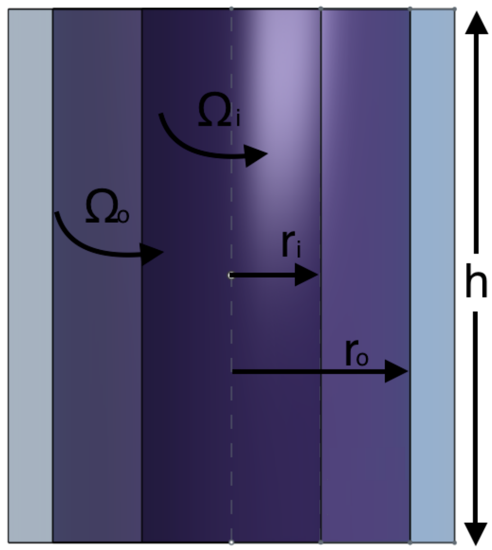

Figure 2.1: An example of circular Couette flow within a basic Taylor Couette system. h is the height of the system, r_i is the radius of the inner cylinder, r_o is the radius of the outer cylinder, and Ω_i and Ω_o are the angular velocities of the inner and outer cylinders, respectively. The darkest purple represents the inner cylinder while the light blue represents the outer cylinder wall.

2.2 Navier-Stokes Equation and Reynolds Numbers

The movement of viscous fluids is governed by the Navier-Stokes equations. Because our flow is incompressible, the incompressible form of the Navier-Stokes equation is used to find the Reynolds number. The Reynolds number is derived by non-dimensionalizing the incompressible Navier Stokes equation [Tag94]

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}, \tag{2.1}$$

where $\mathbf{u}$ is a velocity field, p is a pressure field, and the Reynolds number is Re. The non-dimensional Reynolds number is expressed as [Tag94]

$$Re = \frac{r\Omega d}{v}, \tag{2.2}$$

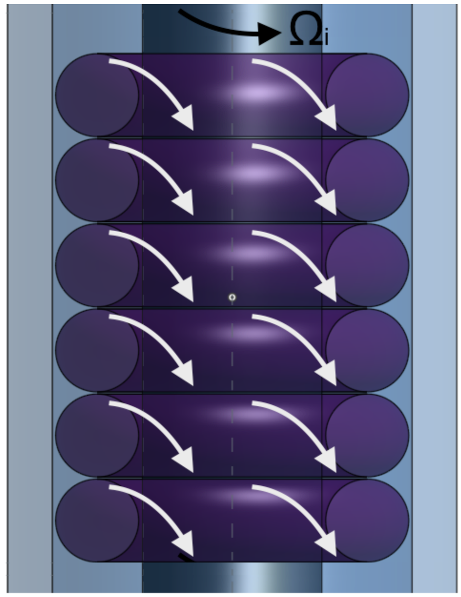

Figure 2.2: An example of Taylor vortex flow within a basic Taylor Couette system. Ω_{i} is the angular velocity of the inner cylinder. Each purple ring represents a vortex layer with the white arrows showing the direction.

where r is the radius of the cylinder, Ω is the cylinder's angular velocity, $d = r_{\mathrm{o}} - r_{\mathrm{i}}$, and ν is the kinematic viscosity of the fluid.

A Taylor Couette system has two Reynolds numbers, as there are two cylinders. Re_{i} is the Reynolds number for the inner cylinder, and Re_{o} is used for the Reynolds number of the outer cylinder. These two terms come from the inner cylinder angular velocity, Ω_{i} and the outer cylinder angular velocity Ω_{o}. Therefore, the two Reynolds numbers, Re_{i} and Re_{o} are

$$Re_{\mathrm{i}} = \frac{r_{\mathrm{i}}\Omega_{\mathrm{i}}d}{v}, \tag{2.3}$$

and

$$Re_{\mathrm{o}} = \frac{r_{\mathrm{o}}\Omega_{\mathrm{o}}d}{v}, \tag{2.4}$$

These definitions are convenient, because as long as the geometry of the system does not change, the Reynolds number roughly corresponds to the non-dimensional linear velocity of the respective cylinder's surface. The Reynolds number is used as a measure of the relative strength of the inertial to viscous forces. So when, the Reynolds number reaches a certain value, the inertial forces overcome the viscous forces and the flow transitions from circular Couette to Taylor vortex [Tag94].

The geometry of the Taylor Couette system can also influence what flow regime is exhibited. Taylor Couette system geometry is h, r_i, and r_o. The quantities that define the system are the two Reynolds numbers from before, Re_i and Re_o, η, and Γ, the aspect ratio which is expressed as $\Gamma = \frac{h}{d}$, where $d = r_\mathrm{o} - r_\mathrm{i}$ [Tag94]. η captures the curvature of the flow while Γ captures the flow's spanwise aspect ratio [BE].

2.3 Primary Instability

As stated before, when the cylinders' rotation rates reach a certain speed the laminar circular Couette flow will transition to a Taylor vortex flow. The primary instability occurs when the Taylor number, and by extension, the inner Reynolds number, reaches a critical value. This is because the equation of the Taylor number is a function of Re_i. When the Taylor number reaches a critical number, which is around 1700 [Kos93], the system undergoes transition. This number is also referred to as the critical Taylor number, Ta_c. The Taylor number is a dimensionless value described by the equation [Tag94]

$$Ta = 4(Re_\mathrm{i})^2 \frac{1-\eta}{1+\eta}\left(1 - \frac{\mu}{\eta^2}\right), \tag{2.5}$$

where η is the ratio of the cylinders radii, μ is the ratio of angular velocities of both cylinders, and Re_i is the inner Reynolds number.

There are two things to note from Eq. 2.5. First, the Taylor number is dependent on the Reynolds number, so when the inner Reynolds number reaches a certain value the Taylor Couette flow will undergo transition. This Reynolds number is known as Re_c or the critical Reynolds number. Second, the Taylor number is dependent on the geometry of the system. In particular, one can see that η influences when the critical Taylor number is reached. This is because the Taylor number is a function of η, μ, and Re. Therefore, the geometry of the system has an effect on when the circular Couette flow will become unstable [Tag94].

The primary instability is indicated by the red line in Fig. 2.3. Early experiments often only rotated one cylinder. This means that only Re_i or Re_o was varied, not both [PBCP94]. When Re_i reaches a certain value, the flow will transition from circular Couette flow, below the red line, to Taylor vortex flow, above the red line [ALS86]. As stated earlier, η and Γ influence the system. This means for different values of η and Γ, the critical value at which the system transitions from circular Couette to Taylor vortex will also change.

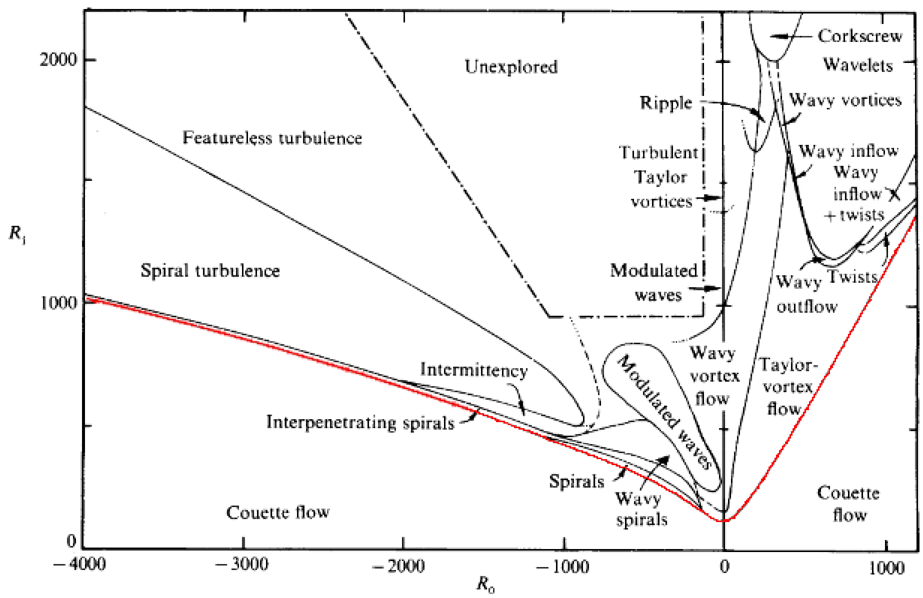

Figure 2.3: Phase diagram shows different flow regimes at varying values for Re_i and Re_o. The horizontal axis is Re_o, and the vertical axis is Re_i. A negative Re_o indicates counter rotation. The primary instability is highlighted by the red line. The system had an η value of 0.883 and a Γ value of 30. This phase diagram is taken from [ALS86].

2.3.1 Primary Instability and the relation to η

The effect η has on the transition to turbulence is relatively untested[1]. Most experiments involving a Taylor Couette system vary the rotation rates of the cylinders as the means to study the system. Rather than changing the geometry of the system multiple times for each experiment, it is far easier to vary the rotation rates of the individual cylinders. If, however, one wanted to change the inner radius, one would have to use different sized inner cylinders for each data set. Similarly, if one wanted to adjust the aspect ratio Γ, one would have to physically adjust the height of both cylinders. In the next chapter we will discuss how we went about increasing η and conducting the experiment.

[1]See [Tay23], [LTK$^+$87], and [MPGL14].

3 Methods

3.1 Preparing the Taylor Couette System

My project was to measure the effect that the geometry of the system has on the transition to turbulence. To accomplish this I needed to be able to adjust the radii and height of the system. Traditionally, this would have not only been incredibly time consuming, but it would have also been very expensive as it would require replacing the cylinders themselves. However, this was accomplished by using 3D printed cylindrical shells. These shells allow the radius of the inner cylinder to be adjusted. However, when we are manipulatingη, the quantity that directly affects Ta, we also affect Γ, the aspect ratio. Γ, however, is something that needs to remain constant even though it does not directly effect Ta. This is because Γ has an effect on the stability of the flow, therefore it needs to remain constant[Avi12]. Looking back on the equation $\Gamma = \frac{h}{d}$. Γ can be kept constant by manipulating the height of the system. Fig. 3.2 shows the Taylor Couette system used in this experiment. In between the two white rings is the test section. These white rings can be moved up and down, and by adjusting them, the height of the system can be set. Through the use of the 3D printed cylindrical shells and by adjusting the distance in between the white rings, η can be manipulated without changing Γ.

A small problem occurred when creating the 3D printed cylindrical shells. The shells had a very rough surface as can be seen on the left part of Fig. 3.1. A jagged surface can influence the flow of the Taylor Couette system. Therefore there was a need to smoothen the surface of the shells. This was done through a process known as acetone finishing. ABS plastic, the plastic the 3D printer uses, melts when exposed to acetone. So cold acetone vapors were used to melt the outer layer of the shell and smoothen it out. The radii of the cylinder before the shell was attached was 47.305 ± 0.005 cm for the inner radius. After attaching the shell the inner cylinder's radius was 51.245 ± 0.005 cm. Finally the outer cylinder's radius was 59.44 ± 0.005 cm.

The fluid used in the system was a combination of shaving cream, glycerine,

Figure 3.1: Picture is of the 3D printed cylindrical shells. An unfinished shell with the rough surface is to the left. On the right is a shell after undergoing acetone finishing. The holes in the center of the shells are there to attach the shell to the inner cylinder.

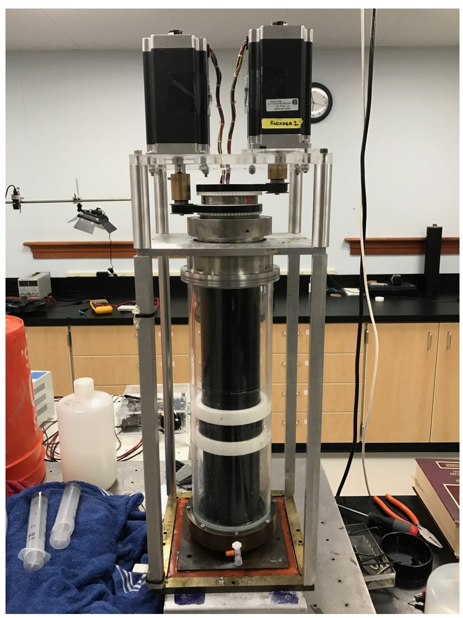

Figure 3.2: The Taylor Couette apparatus used in this thesis. The inner cylinder was built by previous summer research students working for Professor Borrero. The rest was built at the University of Texas at Austin. The outer cylinder is clear and the inner cylinder is black to increase the visibility of the flow in between the two cylinders. The test section is located between the two white rings, the distance in between the two white rings is the system's height, h.

and water. Shaving cream was used because shaving cream contains reflective particles that make it easier to see the various fluid states produced. The viscosity of the solution was measured through the use of a viscometer. The viscosity was measured at 20° Celsius. The viscometer's constant at 20 degrees Celsius was 0.1405 mm^2/s. The kinematic viscosity measured was 10.89 ± 3 mm^2/s.

The motors, the black boxes on the top of the system shown in Fig. 3.2, are attached to a computer and are controlled by a program created in LabView. Each motor is attached to a cylinder, allowing the program to rotate both the inner and outer cylinders at specific velocities. The motors are able to both co-rotate and counter rotate the cylinders.

3.2 Performing the experiment

The system was fully assembled and is shown in Fig. 3.2, and the prepared fluid filled the gap in the apparatus. Through the use of the LabView program, the outer cylinder was then rotated at various velocity settings.

At each outer cylinder velocity setting, the inner cylinder was first rotated at a speed that resulted in circular Couette flow, the current inner angular velocity became ω_{floor}. After waiting 5 - 10 minutes to confirm that rotating at ω_{floor} resulted in circular Couette flow, the rotation rate was then increased and the system was run at this speed for 5 - 10 minutes. If the flow state was circular Couette flow, then the current inner angular velocity is the new ω_{floor} and the speed was increased again. This process repeated until the system underwent transition to Taylor vortex flow. The transition can be seen by eye as shown in Fig. 3.3. This velocity was noted as ω_{ceil}, and then the inner cylinder rotation rate returned to the ω_{floor}. The system was run at ω_{floor} for 5-10 minutes before we increased the rotation rate to some point in between ω_{ceil} and ω_{floor}. If the system did not undergo transition, then ω_{floor} is now the current inner cylinder velocity. If the system did undergo transition, then the current velocity became the new ω_{ceil} and the velocity of the inner cylinder was slowed down to ω_{floor}, After another 5 - 10 minutes we increased the inner angular velocity again. This process repeats until ω_{ceil} and ω_{floor} were nearly equal, for example $\omega_{\text{ceil}} = 1020$ Pos/s and $\omega_{\text{floor}} = 1030$ Pos/s. Then the median in between ω_{ceil} and ω_{floor} was recorded as the critical inner angular velocity for the corresponding outer angular velocity, for example the critical inner velocity was 3216 Pos/s and the outer velocity was -800 Pos/s. This process was repeated for various outer cylinder angular velocities and each critical inner velocity being recorded in a spreadsheet.

The system was changed by attaching the 3D printed shell to the test zone. The adjusted testing zone can be seen in Fig. 3.4. Then the testing process was repeated once more using the same outer cylinder velocities. The critical inner velocities were recorded in a spreadsheet.

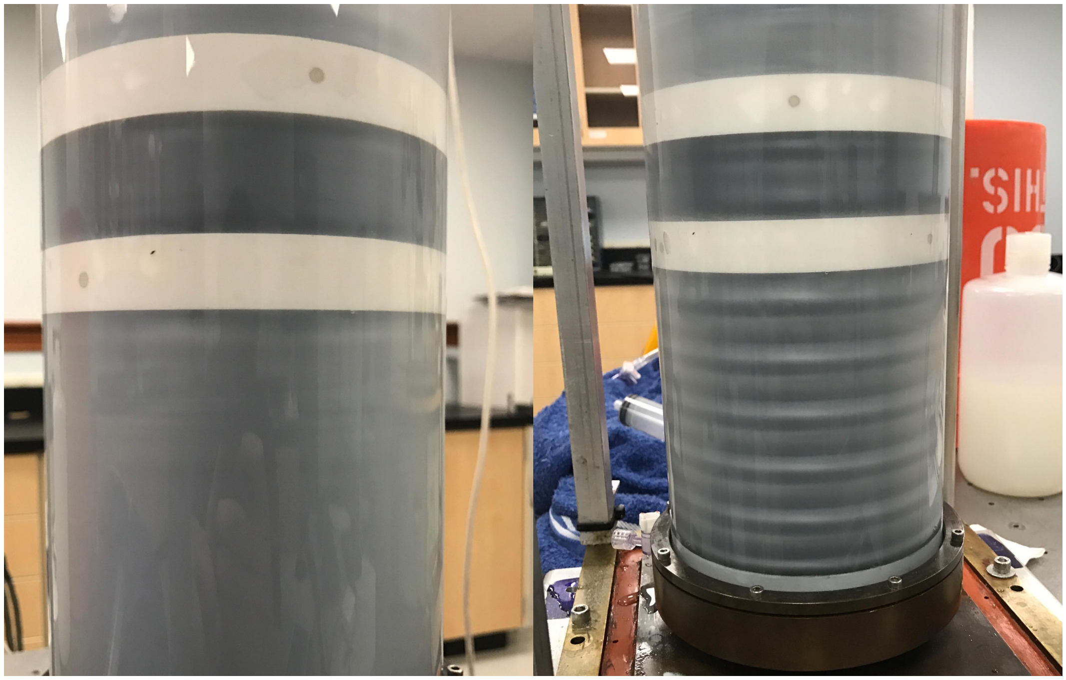

Figure 3.3: Image of what circular Couette flow and Taylor vortex flow look like in the apparatus. The left is circular Couette flow, and the right is Taylor vortex flow.

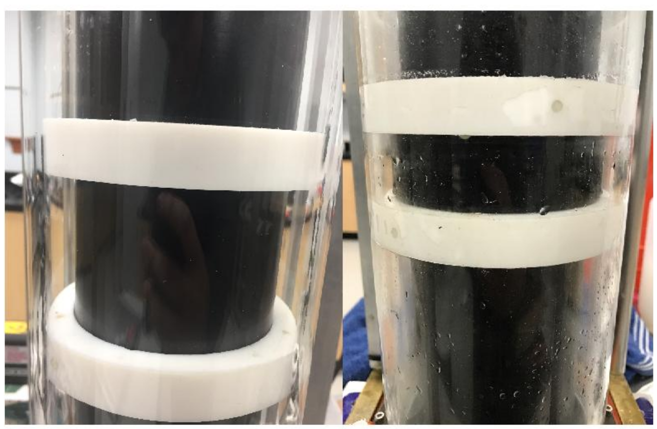

Figure 3.4: Image of the test zone with and without a shell attached. The left side is the test zone without the shell, the right side is the test zone with the shell attached. Notice that the heights are not the same as Γ needs to be kept constant.

4 Results

4.1 Calculation of Re_{ic}

A Matlab program was used to calculate each Re_{ic} using the formula $Re = \frac{r\Omega d}{\nu}$, with r being the inner cylinder's radius (mm), Ω being the inner critical velocity (rad/s), d being the gap between the inner and outer cylinders (mm), and ν being the kinematic viscosity of the solution (mm/s^2). The kinematic viscosity of the fluid used was 10.88 ± 3 mm^2/s. The original r_{i} was 47.30 ± 1 mm. r_{o} was a constant 59.44 ± 0.1 mm. This resulted in an η value of 0.796 for the original system. The Γ of the original system was 4.97. After the shell was attached and the system was adjusted accordingly, r_{i} became 51.24 ± 1 mm. This resulted in an η value of 0.862 and a Γ of 4.19. The Γ difference is from some problems encountered in the acetone finishing process that made the shell's height shorten. The results of the Re_{ic} calculations are summarized in Fig. 4.1.

4.2 Implications

Figure 4.1 indicates that the manipulation of η will cause the primary instability's corresponding critical Reynolds number to increase or decrease. This is dependent on whether η was increased or decreased. If η was increased, Re_{ic} should increase as well. Similarly if η decreases, Re_{ic} should decrease as well. In addition if compared to Fig. 2.3 where the η value was 0.883 and Γ was 30, the shape of curves in Fig. 4.1 match the corresponding part, where Re_{o} is near zero, of the primary instability curve. Both have similar shapes to a parabola. The curve for $\eta = 0.862$ however, seems to indicate that changing η values does have some effect on the slope of the curve. However, the main difference seems to be that changing η values affect whether Re_{ic} increases or decreases.

This agrees with the Taylor number equation, Eq. 2.5 because if η increases, both $\frac{1-\eta}{1+\eta}$ and $1 - \frac{\mu}{\eta^2}$ decrease. This means that for Ta_{c} to stay the same, Re_{ic} must increase to compensate.

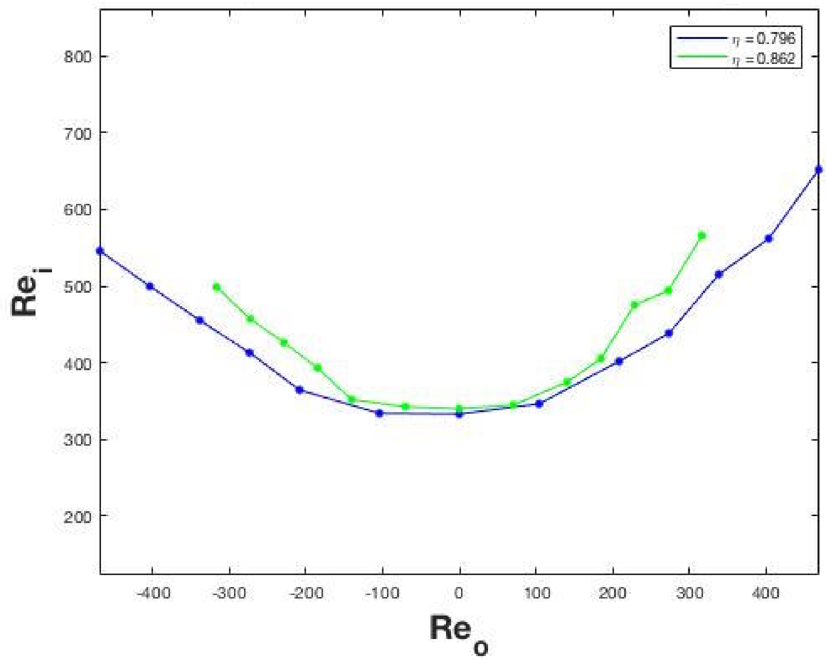

Figure 4.1: Critical Reynolds numbers for different η values. This graph depicts the Re_{o} and their corresponding Re_{ic} for both $\eta = 0.796$ and $\eta = 0.862$.

5 Discussion

5.1 Replication and further research

Adding more sets of data with different η values is the next step for this research. The main problem with this research was it took too much time to create the 3D printed shells. We only managed to get one workable shell out of 16 tries. The 3D printer can take around four hours to print one cylindrical shell. However, the main time sink occurred during the acetone finishing process, which could take up to three days at the longest.

The original reason acetone finishing was used was to smoothen the rough surface of the shells. However, while this process does smoothen out the surface of the shells, it also creates small blemishes. After acetone finishing, the surface of the shell would often have small holes, uneven surfaces, or even bubbles. When these problems occurred, we had to reprint the shell and acetone finish the shell again. In addition, the shells were printed so that they would have a tight fit around the inner cylinder, so that the shell would not be lopsided when attached. Often times, the shells would fit tightly, to inner cylinder before finishing, but after finishing they would no longer fit, which would cause us to have to create another shell. It is because of this that we have come to the conclusion that 3D printed shell, while a cheap and fast way to replicate changes to the inner cylinder, are not a great way to increase the radius of the inner cylinder. For better accuracy specially crafted exchangeable inner cylinders would be a better option; however, these would be far more expensive.

5.2 Smaller η values

There was a severe limitation to how we could manipulate the η values in this experiment because we were attaching a shell to the inner cylinder. The η values we could manipulate could only be equal to 0.796 or greater. So further experimentation could be done on a system with lower η values than 0.796. From this experiment's results it would seem to indicate that increasing the η values would also increase Re_c but this my not be true for η values less than 0.796. This requires further experimentation.

5.3 Kinematic Viscosity

Kinematic viscosity is a constant used in calculations for the Reynolds numbers. However, kinematic viscosity does actually change in response to temperature. As the Taylor Couette system currently has no way to regulate the temperature of the fluid, the viscosity could be changing and creating noisy data. If some way to regulate the temperature of the fluid was found, tests could be done with constant temperatures. Tests could even be done at different temperatures which may also generate interesting data.

6 Conclusion

The primary instability of the Taylor Couette system is the point that the smooth laminar circular Couette flow transitions into the Taylor vortex flow. The primary instability occurs when $Ta > Ta_{\mathrm{c}}$, and because Ta is a function of Re, the primary instability also occurs when $Re > Re_{\mathrm{c}}$. We studied a Taylor Couette system before and after η increased through the used of 3D printed cylindrical shells. These shells attach to the inner cylinder, increasing the inner cylinders radius, thereby increasing η. Through the use of these shells we found that when η is increased, Re_{c} also increases. We believe this is to compensate for both the $\frac{1-\eta}{1+\eta}$ and the $1 - \frac{\mu}{\eta^2}$ terms in the equation for Ta decreasing. This general trend is depicted in the curves shown in Fig. 4.1.

However, the data samples taken within this experiment are small because the shells used to increase η took too long to make. In addition to taking a long time to create the shells, the data are likely slightly inaccurate because the process used to finish the shells was extremely inconsistent, which may have made the surface of the shells uneven. Thus, further testing is needed, both for increasing η and decreasing it. Therefore, while it may take longer and be more expensive to create, we would suggest having specially made inner exchangeable cylinders. These cylinders will have a greater chance of having more consistent and accurate data than the hastily made shells of inferior quality. With better and more thorough data we may be able to gain a greater understanding on when the primary instability in a Taylor Couette system will occur which will greatly increase our knowledge about fluid dynamics.

Bibliography

[ALS86] C.D. Andereck, S.S. Liu, and H.L. Swinney, *Flow regimes in a circular Couette system with independently rotating cylinders*, J. Fluid Mech. **164** (1986), 155–183.

[Avi12] M. Avila, *Stability and angular-momentum transport of fluid flows between corotating cylinders*, Phys. Rev. Lett. **108** (2012).

[BC12] M.J. Burin and C.J. Czarnocki, *Subcritical transition and spiral turbulence in circular Couette flow*, J. Fluid Mech **709** (2012), 106–122.

[BE] D. Borrero-Echeverry, *Subcritical transition to turbulence in Taylor-Couette flow*, Ph.D. Dissertation, Georgia Institute of Techonology, year=2014.

[Cha02] S.J. Chapman, *Subcritical transition in channel flows*, J. Fluid Mech **451** (2002), 35–97.

[CKA07] I.M. Cohen, P.K. Kundu, and P.S. Ayyaswamy, *Fluid dynamics*, Academic Press, 2007.

[Eck07] B. Eckhardt, *Turbulence transition in pipe flow: some open questions*, Nonlinearity **21.1** (2007), T1–T11.

[Kos93] E.L. Koschmieder, *Bernard Cells and Taylor Vortices*, Cambridge University Press (1993).

[LTK+87] W.F. Langford, R. Tagg, E.J. Kostelich, H.L. Swinney, and M. Folubitsky, *Primary instabilities and bicriticality in flow between counter-rotating cylinders*, Phys. Fluids **31** (1987), 766–785.

[Mal88] A. Mallock, *Determination of the viscosity of water*, Proc. R. Soc. London **45** (1888), 126–132.

[MPGL14] N.A. Moussa, S. Poncet, A. Ghezal, and M. Lounis, *Effect of radius ratio on the stability of co- and counter- Taylor Couette flows*, Proceedings of the International Conference on Heat Transfer and Fluid Flow (2014), 1–5.

[PBCP94] J.M. Piau, M. Bremond, J.M. Couette, and M. Piau, *Maurice Couette, one of the founders of rheology*, Rheologica Acta **33** (1994), 357–368.

[Tag94] R. Tagg, *The Couette-Taylor problem*, Nonlinear Science Today **4** (1994), 1–25.

[Tay23] G.I. Taylor, *Stability of a viscous liquid contained between two rotating cylinders*, Philosophical Transactions of the Royal Society of London. **22** (1923), 289–343.