CS343

Kleinberg and Tardos, Chapter 2, problem 6: Find the number of additions executed. Note, the summation below actually counts the number of terms added as opposed to the number of additions, e.g. $A_1 + A_2$ has 2 terms but one addition. However, the complexity will be the same and, besides, the exercise is more about doing summations.

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=i}^{j} 1 &= \sum_{i=1}^{n}\sum_{j=i+1}^{n} j - i + 1 \\
&= \sum_{i=1}^{n}\left[\left(\sum_{j=i+1}^{n} j\right) + \left((1-i)\sum_{j=i+1}^{n} 1\right)\right] \\
&= \sum_{i=1}^{n}\left(\frac{n(n+1)}{2} - \frac{i(i+1)}{2} + (1-i)(n-i)\right)
\end{aligned}$$

After lots of simplification (just algebra!), we get

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=i}^{j} 1 &= \sum_{i=1}^{n}\left(\frac{i^2}{2} - (n+3/2)i + \frac{n(n+3)}{2}\right) \\
&= \left(\frac{1}{2}\sum_{i=1}^{n} i^2\right) - \left((n+3/2)\sum_{i=1}^{n} i\right) + \left(\frac{n(n+3)}{2}\sum_{i=1}^{n} 1\right) \\
&= \left(\frac{1}{2}\frac{n(n+1)(2n+1)}{6}\right) - \left((n+3/2)\frac{n(n+1)}{2}\right) + \left(\frac{n(n+3)}{2}n\right)
\end{aligned}$$

Lots more simplification gives the final result ...

$$\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=i}^{j} 1 = \frac{n}{6}\left(n^2 + 3n - 4\right)$$

To check the result, compare the value calculated directly from the summation with the value calculated from the above equation to see if they match. For example, when n = 3, we should get 7 (do you get this?)