Complexity Notation

O **notation:** We say that a function $f(n)$ has a time complexity $O(g(n))$ if there exist constants $c > 0$ and n_0 such that

$$0 \leq f(n) \leq cg(n) \text{ for } n \geq n_0$$

Ω **notation:** We say that a function $f(n)$ has a time complexity $\Omega(g(n))$ if there exist positive constants c and n_0 such that

$$0 \leq cg(n) \leq f(n) \text{ for } n \geq n_0$$

Θ **notation:** We say that a function $f(n)$ has a time complexity $\Theta(g(n))$ if there exist positive constants c_1 , c_2, and n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for } n \geq n_0$$

o **notation:** We say that a function $f(n)$ has a time complexity $o(g(n))$ if for *any* positive constant c, there exists an $n_0 > 0$ such that

$$0 \leq f(n) < cg(n) \text{ for } n \geq n_0$$

ω **notation:** We say that a function $f(n)$ has a time complexity $\omega(g(n))$ if for *any* positive constant c, there exists an $n_0 > 0$ such that

$$0 \leq cg(n) < f(n) \text{ for } n \geq n_0$$

Limits

Assume that $f(n)$ is asymptotically non-negative and $g(n)$ is is asymptotically positive.

- $\lim_{n\to\infty} f(n)/g(n) = d$ where $0 < d < \infty$ implies
  $f(n) = O(g(n))$, $g(n) = O(f(n))$, and $f = \Theta(g)$
- $\lim_{n\to\infty} f(n)/g(n) = \infty$ implies
  $g = O(f)$, $f(n) \neq O(g(n))$, $f = \Omega(g)$, and $f \neq \Theta(g)$
- $\lim_{n\to\infty} f(n)/g(n) = 0$ implies
  $f(n) = O(g(n))$, $g(n) \neq O(f(n))$, and $f \neq \Theta(g)$
- $f = o(g(n))$ if and only if $\lim_{n\to\infty} f(n)/g(n) = 0$
- $f = \omega(g(n))$ if and only if $\lim_{n\to\infty} f(n)/g(n) = \infty$
- Recall L'Hopital's Rule: $\lim_{n\to\infty} f(n)/g(n) = \lim_{n\to\infty} f'(n)/g'(n)$
- Warning: Don't always assume the converse. For example, $f(n) = O(g(n))$ does not necessarily imply that $\lim_{n\to\infty} f(n)/g(n) = d$. Counterexample: let $f(n) = n$ and $g(n) = n(1 + \sin n)$. In this case, the limit doesn't exist.