Name: __________

CS443: Analysis of Algorithms Final, Sp 03

Score:	1.	(max 5)	6.	(max 5)
	2.	(max 10)	7.	(max 15)
	3.	(max 5)	8.	(max 20)
	4.	(max 10)	9.	(max 10)
	5.	(max 20)	Xtra.	(max 10)
	Total:		(max 100 + 10 xtra)	

This exam is closed book. Calculators are not allowed.

1. (5 pts) Carefully state the definition of $f(n) = O(g(n))$.

2. (10 pts) Use the definition you gave in the previous problem to prove or disprove the claim: $2^{n+1} = O(2^n)$. Carefully explain all of your steps.

3. (5 pts) What is an AVL tree?

4. (10 pts) Use the method of telescoping to solve the following recurrence.

$$T(n) = T(n-1) + n \text{ with } T(1) = 1$$

5. (4 pts each, 20 pts total) State whether the Master Theorem (see last page) applies to the following recurrences. In all cases, assume that $T(1) = c$ for some $c > 0$. If the Master Theorem does apply, use it to find the Θ complexity of the recurrence. Specifically state which case applies. If the Master Theorem does not apply, state why it does not apply but do not solve the recurrence.

   (a) $T(n) = 2\,T(\frac{n}{4}) + \sqrt{n}$

   Does the MT apply? Circle: Yes or No

   If yes, the case is: __________

   and $T(n) = \Theta($ $)$

   (b) $T(n) = 3\,T\left(\frac{n}{4}\right) + n^2$

   Does the MT apply? Circle: Yes or No

   If yes, the case is: __________

   and $T(n) = \Theta($ $)$

(c) $T(n) = 6n\,T(n/3) + 2n^2$

Does the MT apply? Circle: Yes or No

If yes, the case is: __________

and $T(n) = \Theta($ $\qquad$ $)$

(d) $T(n) = 2\,T(\sqrt{n}) + n$

Does the MT apply? Circle: Yes or No

If yes, the case is: __________

and $T(n) = \Theta($ $\qquad$ $)$

(e) $T(n) = 25\,T\left(\frac{n}{5}\right) + n^2$

Does the MT apply? Circle: Yes or No

If yes, the case is: __________

and $T(n) = \Theta($ $\qquad$ $)$

6. (5 pts) What characteristics of a problem do you look for when considering whether or not to use dynamic programming?

7. (15 pts total) Pick your favorite dynamic programming problem out of the following: Matrix chain, LCS, 0-1 Knapsack, or Pretty printing:

   (a) (5 pts) Describe the problem.

(b) (5 pts) How were the subproblems defined?

(c) (5 pts) What recursion was used?

8. (20 pts) The Fractional Knapsack Problem.

   (a) (10 pts) Describe a greedy algorithm for loading your knapsack so as to maximize the benefit.

   (b) (10 pts) Prove that this algorithm satisfies the Greedy Choice Property.

9. (10 pts) In class we proposed a greedy approach for solving the Cheapest Path problem that did not result in optimal solution. Does this greedy approach to the Cheapest Path satisfy the Greedy Choice Property? If yes, show the proof. If not, show how the proof falls apart.

10. (10 pts) Extra Credit: Prove that the regions formed by n (possibly overlapping) circles in the plane can be colored with two colors such that any neighboring regions are colored differently. (Note, this problem does not work if the circles are replaced with rectangles, so your proof must make use of some property of circles.) *Credit will not be given unless your solution is almost completely correct.*

The Master Theorem

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the non-negative integers by the recurrence

$$T(n) = \begin{cases} c & \text{if } n < n_0 \\ aT(n/b) + f(n) & \text{if } n \geq n_0 \end{cases}$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ can be bounded asymptotically as follows:

1. If $f(n) = O(n^{log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{log_b a} \log^k n)$ for some $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \log_2^{k+1} n)$.
3. If $f(n) = \Omega(n^{log_b a + \epsilon})$ for some constant $\epsilon > 0$, and $af(n/b) \leq \delta f(n)$ for some $\delta < 1$ and $n \geq n_0$, then $T(n) = \Theta(f(n))$.