Physically Based Modeling

Objects obey physical laws, e.g. gravity, collisions, spring forces, etc.

Particle System

Each particle:

- position moves over time based on the forces acting on it, i.e. it obeys $f = ma$.
- has 6 degrees of freedom: 3 position, 3 velocity

Equations of Motion:

position: $p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

velocity: $v = \dot{p} = \frac{dp}{dt} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$

These 6 degrees of freedom are combined into a single vector u referred to as the *phase space*:

$$u = \begin{pmatrix} p \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

and where

$$\dot{u} = \begin{pmatrix} \dot{p} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ f/m \end{pmatrix}$$

describes the path of the particle over time. Note,

$$\dot{v} = \text{acceleration} = a = f/m$$

An equation of the form

$$\dot{u} = h(u, t)$$

where h is some function, is referred to as a *1st order differential equation.* If it can't be solved exactly, then we solve it numerically.

Taylor's Expansion

Taylor's expansion says that

$$u(t_0 + \Delta t) = u(t_0) + \Delta t\ \dot{u}(t_0) + \frac{(\Delta t)^2}{2!}\ddot{u}(t_0) + ...$$

It is exact but requires summing an infinite number of terms. For small Δt we can approximate using *Euler's Method*

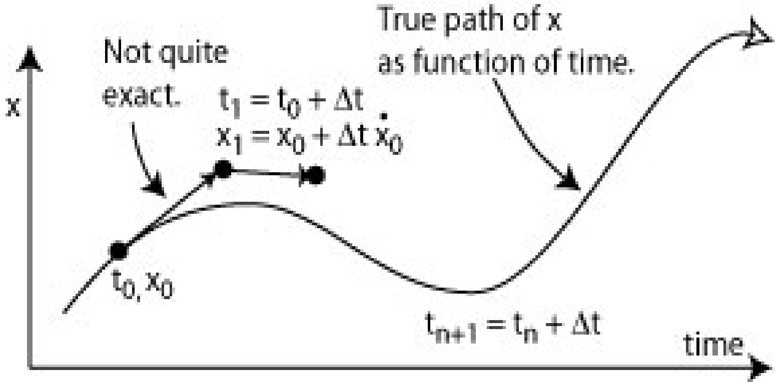
$$u(t_0 + \Delta t) = u(t_0) + \Delta t\ h(u_0, t_0) + \Theta((\Delta t)^2)$$

Dropping the last term gives

$$u(t_0 + \Delta t) = u(t_0) + \Delta t\ h(u_0, t_0)$$

This is iterated to obtain its value u at t_0, $t_0 + \Delta t$, $t_0 + 2\Delta t$, ... Often this is written as

$$u_n = u_{n-1} + \Delta t\ h(u_{n-1}, t_{n-1})$$



Euler's method is the simplest method for approximating differential equations. A much better approximation is called the *Midpoint* or *Runge Kutta Method* given by

$$\begin{aligned} u_{n+1} &= u_n + k_2 + \Theta((\Delta t)^3) \\ k_2 &= \Delta t\ h(u_n + \frac{1}{2}k_1, t_n + \frac{1}{2}\ \Delta t) \\ k_1 &= \Delta t\ h(u_n, t_n) \end{aligned}$$

Examples

1. Constant Motion

   Assume that there are no forces, $f = 0$, so that $a = 0$ and $v = v_c =$ constant. Then

$$\dot{u} = \begin{pmatrix} \dot{p} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v_c \\ 0 \end{pmatrix}$$

Since $\dot{p} = v_c$ is constant, we have that $\ddot{p} = \dddot{p} = ... = 0$. Putting this into Taylor's Expansion gives

$$u(t_0 + \Delta t) = u(t_0) + \Delta t \; h(u_0, t_0) = u(t_0) + \Delta t \begin{pmatrix} v_c \\ 0 \end{pmatrix}$$

or

$$p(t_0 + \Delta t) = p(t_0) + \Delta t \; v_c \tag{1}$$

which is what one expects for constant velocity. This is exact - no approximation is required.

Iterating, we have

$$u_n = \begin{pmatrix} p_n \\ v_n \end{pmatrix} = \begin{pmatrix} p_{n-1} \\ v_c \end{pmatrix} + \Delta t \begin{pmatrix} v_{c,x} \\ v_{c,y} \\ v_{c,z} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p_0 \\ v_c \end{pmatrix} + n \; \Delta t \begin{pmatrix} v_{c,x} \\ v_{c,y} \\ v_{c,z} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Although iterating is hardly necessary since equation (1) is so simple.

2. Gravity

   Assume the force is a constant in negative y-direction

$$a = \dot{v} = \frac{f}{m} = \begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix}$$

so that

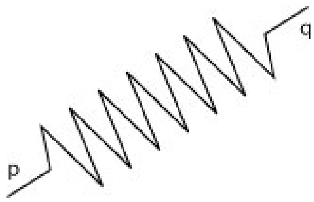
$$\dot{u} = \begin{pmatrix} v \\ \frac{f}{m} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ 0 \\ -g \\ 0 \end{pmatrix}$$

Applying Euler's Method, gives

$$u_n = u_{n-1} + \Delta t \; h(u_{n-1}, t_{n-1})$$

$$u_n = \begin{pmatrix} p_n \\ v_n \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \\ v_{x,n-1} \\ v_{y,n-1} \\ v_{z,n-1} \end{pmatrix} + \Delta t \begin{pmatrix} v_{x,n-1} \\ v_{y,n-1} \\ v_{z,n-1} \\ 0 \\ -g \\ 0 \end{pmatrix}$$

3. Springs



Hooke's Law gives

$$f = -k_s(|d| - s)\frac{d}{|d|}$$

where

$$\begin{aligned} k_s &= \text{ spring constant} \\ d &= p - q = \text{ direction of force} \\ s &= \text{ spring resting length} \end{aligned}$$

So we have

$$a = \dot{v} = \frac{f}{m} = -\frac{k_s}{m}(|d| - s)\frac{d}{|d|}$$

and

$$\dot{u} = \begin{pmatrix} v \\ \frac{f}{m} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \frac{f_x}{m} \\ \frac{f_y}{m} \\ \frac{f_y}{m} \end{pmatrix}$$

Applying Euler's Method, gives

$$u_n = u_{n-1} + \Delta t \; h(u_{n-1}, t_{n-1})$$

$$u_n = \begin{pmatrix} p_n \\ v_n \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \\ v_{x,n-1} \\ v_{y,n-1} \\ v_{z,n-1} \end{pmatrix} + \Delta t \begin{pmatrix} v_{x,n-1} \\ v_{y,n-1} \\ v_{z,n-1} \\ \frac{f_x}{m} \\ \frac{f_y}{m} \\ \frac{f_y}{m} \end{pmatrix}$$