Math 142 Review Problems and Solutions

The following list includes example problems with solutions worked out step by step. They are in no particular order, but you may find many of them helpful in your review of Calculus I topics.

1. **Finding a tangent line.** Find the equation of the line tangent to the curve $y = \sqrt{x + 3x^2}$ at the point $(1, 2)$.

   Solution: We will use the point-slope equation for a line. The equation of the line which passes through the point (x_0, y_0) and has slope m is given by

   $$y - y_0 = m(x - x_0).$$

   We find the slope of this line using the derivative y' which is calculated using the chain rule.

   $$y' = \frac{d}{dx}(\sqrt{x + 3x^2}) = \frac{1}{2}(x + 3x^2)^{-1/2}(1 + 6x)$$

   To find the slope at the point $(1, 2)$ we evaluate the derivative at $x = 1$ which gives $m = \frac{1}{2}(1 + 3 \cdot 1^2)^{-1/2}(1 + 6 \cdot 1) = \frac{7}{4}$. Therefore the equation of the tangent line is

   $$y - 2 = \frac{7}{4}(x - 1).$$

2. **Graph sketching.** Use the first and second derivatives to sketch the graph of the function $f(x) = x^4 - 6x^2$.

   Solution. Recall, the first derivative can be used to determine intervals where the function is increasing/decreasing.

   If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.
   If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.

   And the second derivative can be used to determine intervals where the function in concave up/down.

   If $f''(x) > 0$ on an interval, then $f(x)$ is concave up on that interval.
   If $f''(x) < 0$ on an interval, then $f(x)$ is concave down on that interval.

   We calculate the first and second derivative of the given function. $f'(x) = 4x^3 - 12x$ and $f''(x) = 12x^2 - 12$.

   To find the intervals over which $f'(x)$ is positive and negative we find the zeros of $f'(x)$ test x-values between these zeros. The values where $f'(x)$ is zero or undefined are called the *critical values* of $f(x)$.

   Setting $f'(x) = 4x^3 - 12x = 4x(x^2 - 3) = 0$, the critical values are $x = 0$ and $x = \pm\sqrt{3}$. Below is a number line with the critical points labeled. A plus or minus sign over each interval denotes whether the first derivative is positive or negative on each interval. The decr/incr below each interval denotes that the function f is decreasing/increasing on that interval.

   $$\underset{f\text{ decr}}{\overset{-}{\longrightarrow}} \ (-\sqrt{3}) \ \underset{f\text{ incr}}{\overset{+}{\longrightarrow}} \ 0 \ \underset{f\text{ decr}}{\overset{-}{\longrightarrow}} \ \sqrt{3} \ \underset{f\text{ incr}}{\overset{+}{\longrightarrow}}$$

Next we determine the intervals over which the function is concave up/down by finding the values where $f''(x)$ is zero or undefined. $f''(x) = 12x^2 - 12 = 12(x^2-1) = 12(x-1)(x+1) = 0$ so the critical values are $x = \pm 1$. Below is a number line with the critical points labeled. A plus or minus sign over each interval denotes whether the second derivative is positive or negative on each interval. The C.U./C.D. below each interval denotes whether the function f is concave up/down on that interval.

+		−		+
C.U.	-1	*C.D.*	1	*C.U.*

Putting this information together we can sketch the graph of the function $f(x) = x^4 - 6x^2$. [graph omitted. try graphing it and check your answer on your calculator.]

3. **Implicit Differentiation.** Find the equation of the line tangent to the curve below at the point (3,3).

$$x^3 + y^3 = 6xy$$

Solution. Implicit differentiation requires imagining y is a function of x and using the appropriate differentiation rules.

$$\begin{aligned} \frac{d}{dx}(x^3+y^3) &= \frac{d}{dx}(6xy) \\ 3x^2 + 3y^2 \cdot \frac{dy}{dx} &= 6 \cdot y + \frac{dy}{dx}(6x) \\ 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} &= 6y - 3x^2 \\ \frac{dy}{dx}(3y^2 - 6x) &= 6y - 3x^2 \\ \frac{dy}{dx} &= \frac{6y-3x^2}{3y^2-6x} \end{aligned}$$

So at the point (3, 3) the slope of the tangent line is $m = \frac{6\cdot 3 - 3\cdot 3^2}{3\cdot 3^2 - 6\cdot 3} = -1$. Thus the line tangent to the given curve at the point (3, 3) is

$$y - 3 = -1(x-3).$$

4. **Definite Integral** If f is a continuous function on the interval $[a,b]$, we divide the interval into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. The endpoints of these subintervals are denoted $x_0 = a, x_1, x_2, \ldots x_n = b$ and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the interval $[x_{i-1}, x_i]$. (Typically we will consider the case when x_i^* is the right hand or left hand endpoint of each subinterval.) Then the **definite integral of** f **from** a **to** b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\Delta x\right]$$

Vocabulary and remarks. The quantity $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a **Riemann sum** for $f(x)$ and is sometimes denoted by R_n.

The **lower limit** of integration is a and the **upper limit** of integration is b.

The integral only calculates an area if $f(x)$ is positive on the interval $[a, b]$. For functions $f(x)$ which are not always positive on the interval $[a, b]$, then the integral calculates the **net area**, that is

$$\int_a^b f(x)\, dx = (\text{area above x-axis and below } f(x)) - (\text{area below x-axis and above } f(x))$$

Use the definition of the integral to calculate $\displaystyle\int_1^5 3x - 4\, dx$ using right hand endpoints for the sample points in the Riemann sum.

Solution. We begin by finding expressions for Δx, x_i, and $f(x_i)$. (note: using right hand endpoints implies $x_i^* = x_i = a + i(\Delta x)$.)
$\Delta x = \frac{b-a}{n} = \frac{5-1}{n} = \frac{4}{n}$,
$x_i = a + i\Delta x = 1 + i\frac{4}{n}$,
$f(x_i) = 3(1 + i\frac{4}{n}) - 4$.

Next, we plug the expressions above into the equation for the Riemann sum and simplify to an expression without summation notation. We will be using the formulas $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{i=1}^{n} 1 = n$ from Appendix E of your textbook.

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n} \left(3(1 + i\frac{4}{n}) - 4\right)\left(\frac{4}{n}\right) \\
&= \left(\frac{4}{n}\right)\sum_{i=1}^{n} \left(3(1 + i\frac{4}{n}) - 4\right) \\
&= \left(\frac{4}{n}\right)\sum_{i=1}^{n} \left(3 + i(\frac{12}{n}) - 4\right) \\
&= \left(\frac{4}{n}\right)\sum_{i=1}^{n} \left(-1 + i(\frac{12}{n})\right) \\
&= \left(\frac{4}{n}\right)\left(\sum_{i=1}^{n} -1 + \sum_{i=1}^{n} i(\frac{12}{n})\right) \\
&= \left(\frac{4}{n}\right)\left(-\sum_{i=1}^{n} 1 + (\frac{12}{n})\sum_{i=1}^{n} i\right) \\
&= \left(\frac{4}{n}\right)\left(-n + (\frac{12}{n})\left(\frac{n(n+1)}{2}\right)\right) \\
&= \left(-4 + (\frac{48}{n^2})\left(\frac{n(n+1)}{2}\right)\right) \\
&= \left(-4 + (24)\left(\frac{(n+1)}{n}\right)\right)
\end{aligned}$$

Last we take the limit as n goes to infinity of the Riemann sum expression above.

$$\int_1^5 3x - 4\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\Delta x\right] = \lim_{n\to\infty}\left(-4 + (24)\left(\frac{(n+1)}{n}\right)\right) = -4 + 24 = 20$$

5. **Recognizing Riemann Sums as Integrals.** The following expression is a limit of Riemann sum using right hand endpoints for the sample points. We want to recognize it as a Riemann sum and evaluate the integral using the fact that integrals calculate **net area**.

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\sqrt{100-\left(-10+i\left(\frac{10}{n}\right)\right)^2}\left(\frac{10}{n}\right)\right]$$

Solution. First, a remark that there are many correct answers to this problem and what follows is one of them. The first thing to observe is the fraction $\frac{10}{n}$ on the right of the expression above. If the expression above is of the form $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$ then the quantity $\frac{10}{n}$ must be the Δx. This means that $\sqrt{100-\left(-10+i\left(\frac{10}{n}\right)\right)^2}$ must be the $f(x_i)$. Next we want to determine which part of this expression is the x_i and which is the $f(x)$. (At this point different correct answers are possible.) Looking at the expression

$$\sqrt{100-\left(-10+i\left(\frac{10}{n}\right)\right)^2}$$

I see the function $\sqrt{100-(stuff)^2}$, so one solution is $x_i = stuff = -10 + i\left(\frac{10}{n}\right)$ and $f(x) = \sqrt{100-x^2}$. Since we are using right hand endpoints in the Riemann sum $x_i = a + i\Delta x = -10 + i\left(\frac{10}{n}\right)$. Thus $a = -10$. Since $\Delta x = \frac{b-a}{n} = \frac{b-(-10)}{n} = \frac{10}{n}$ it must be the case that $b = 0$.

Thus,

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\sqrt{100-\left(-10+i\left(\frac{10}{n}\right)\right)^2}\left(\frac{10}{n}\right)\right] = \int_{-10}^{0}\sqrt{100-x^2}\,dx$$

Now, with a little graphing this integral can be evaluated using high school geometry. The curve $y = \sqrt{100-x^2}$ is the top half of the circle of radius 10 centered at the origin. (graph it!) Integrating from -10 to 0 calculates the area of the quarter of this circle which is in the 2nd quadrant. Thus

$$\int_{-10}^{0}\sqrt{100-x^2}\,dx = \frac{1}{4}\pi(10)^2.$$