MATH 162 Final Exam
December 16, 2002

NAME:____________________________

Circle Instructor Name: Johnson Minock

- **No calculators are allowed on this exam.**
- **Answers such as** $\frac{23 \cdot 5}{30} - \frac{2^5}{3 \cdot 34}$ **are perfectly fine!! However you MUST simplify expressions such as** $\sin(\pi/3)$**.**
- **Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown.**
- **Please include all information about the u-substitutions or integration by parts choice(s) that you make.**

$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$	$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$	$\sin(2x) = 2\sin(x)\cos(x)$

Expression	Substitution	
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,$	$-\pi/2 \le \theta \le \pi/2$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,$	$-\pi/2 < \theta < \pi/2$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,$	$0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$

$L = \int_{t_0}^{t_1} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$	$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$

Part 1		
Problem	Points	Score
1	30	
2	8	
3	18	
4	12	
5	14	
6	9	
7	9	

Part 2		
Problem	Points	Score
8	16	
9	10	
10	22	
11	20	
12	8	
13	12	
14	12	

1. Integrals. [30pts, 6 each.]

Evaluate the integrals. Show all work and include all information about substitutions and integration by parts choices, etc.

(a) $\int_0^{\frac{\pi}{2}} \sin^{86}(x)\cos(x)dx$

(b) $\int_2^{\infty} \frac{1}{x(\ln(x))^2}dx$

(c) $\int \tan^2(x)\sec^4(x)dx$

(d) $\int x^3 e^{x^2}\,dx$

(e) $\int_1^3 \frac{1}{(x-2)^2}dx.$

2. Parametric equations and Area. [8pts]

The ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

can be parameterized by $x = a\cos(t)$ and $y = b\sin(t)$.

Use this parameterization to find the area of an ellipse. Your answer should be a formula in terms if a and b.

3. Volume. [18pts, 3 each]

Consider the region, R which is bounded between the curves

$$\begin{aligned} y &= 4x - x^2 \\ y &= x \end{aligned}$$

(a) Sketch the region. Label the points of intersection.

(b) Write an integral which represents the area of the region. DO NOT INTEGRATE.

(c) Write an integral for the volume of the solid formed by rotating this region R about the x-axis (that is the line $y = 0$.). DO NOT evaluate the integral.

(d) Write an integral for the volume of the solid formed by rotating this region R about the y-axis (that is the line $x = 0$.). DO NOT evaluate the integral.

(e) Write an integral for the volume of the solid formed by rotating this region R about the line $x = -2$.). DO NOT evaluate the integral.

(f) Now consider the solid whose base is the region R and whose cross-sections above the

4. <u>Parametric equations, curve sketching.</u> [12pts]

a) For the parametric curve $x = te^t$, $y = t^2 - 2t$ find all points (x, y) where the tangent is horizontal, and all points (x, y) where the tangent line is vertical.

b) Find the interval(s) of t-values where the curve is increasing, and the interval(s) of t-values where the curve is decreasing.

c) Calculate the following limits to understand the "end" behavior of the curve. [Here "end" behavior means what happens to the curve as $t \to \infty$ and $t \to -\infty$.]

$$\lim_{t\to\infty} x = \lim_{t\to\infty} te^t =$$

$$\lim_{t\to\infty} y = \lim_{t\to\infty} t^2 - 2t =$$

$$\lim_{t\to-\infty} x = \lim_{t\to-\infty} te^t =$$

$$\lim_{t\to-\infty} y = \lim_{t\to-\infty} t^2 - 2t =$$

d) Use the information above to sketch the curve. Find and label the x and y-intercepts.

5. <u>Polar Curves, Graph matching.</u> [14pts]

Write the letter of the graph that best fits the graph of the given *polar curves* in the space provided. For all curves, $0 \leq \theta \leq 2\pi$.

(i) $r = \theta$ ____________________

(ii) $r = 2\sin(\theta)$ ____________________

(iii) $r = 2\cos(\theta)$ ____________________

(iv) $\theta = \frac{\pi}{4}$ ____________________

(v) $r = \frac{\pi}{4}$ ____________________

(vi) $r = 1 + \cos(\theta)$ ____________________

(vii) $r = 1 + \sin(\theta)$ ____________________

A: B: C:

D: E: F:

G: H: I:

6. Polar Curves, Tangent Lines. [9pts]

a) Show that the slope of the line tangent to the polar cure $r = 2\cos(\theta)$ at $\theta = \frac{\pi}{6}$, is $-\frac{1}{\sqrt{3}}$.

b) Use part a) to find the equation of the line (in x-y coordinates) that is tangent to the polar curve $r = 2\cos(\theta)$ at $\theta = \frac{\pi}{6}$.

7. Polar Curves, Arc Length. [9pts]

The graph of the polar curve $r = \sin(\theta) + \cos(\theta)$ is given below.

a) Find two different descriptions of the point **P** in polar coordinates that satisfy the polar curve $r = \sin(\theta) + \cos(\theta)$.

b) Use the formula for arc length in polar coordinates to find the length of the curve $r = \sin(\theta) + \cos(\theta)$ traversed exactly once. [Note: part a) should help you figure out the limits of integration.]

Final Exam, Part 2

8. Sequences and series. [16pts]

Determine whether the statements are TRUE or FALSE. True means always true, false means sometimes false.

(a) ________________ The **sequence** $\{\frac{3n}{n+1}\}_{n=1}^{\infty}$ converges.

(b) ________________ The sequence $\{1,0,1,0,0,1,0,0,0,1,0,0,0,0,1,\cdots\}$ converges to 0.

(c) ________________ $1-1+1-1+1-1+1-\ldots$ converges to zero.

(d) ________________ If $\sum |a_n|$ converges and $\lim_{n\to\infty} |\frac{a_n}{b_n}| = 1$, then the series $\sum |b_n|$ converges.

(e) ________________ If $\sum a_n$ diverges then $\sum |a_n|$ diverges.

(f) ________________ The series $50+10+2+\frac{2}{5}+\frac{2}{25}+\cdots$ is a geometric series which converges to $\frac{125}{2}$

(g) ________________ If the power series $\sum c_n(x-4)^n$ converges for $x=8$ then the series $\sum c_n(3^n)$ is convergent.

(h) ________________ $\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

9. Power Series, Interval of Convergence. [10pts]

Consider the power series

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{5^n\sqrt{n}}$$

a) Find the radius of convergence for the power series.

b) What is the interval of convergence? Justify your answer by checking the endpoints.

10. Series Tests. [22pts]

Fill in the blanks with the correct statement of each series test. Note, some of the blanks below should be filled in with "the **** test tells us NOTHING!"

The Comparison Test:

Suppose $\sum a_n$ and $\sum b_n$ are series with ______________________ terms.

(i) If $a_n \leq b_n$ and the series $\sum b_n$ is divergent, then __________________________________.

(ii) If $a_n \leq b_n$ and the series $\sum a_n$ is divergent, then __________________________________.

(iii) If $a_n \leq b_n$ and the series $\sum b_n$ is convergent, then __________________________________.

(iv) If $a_n \leq b_n$ and the series $\sum a_n$ is convergent, then __________________________________.

Limit Comparison test:

Suppose $\sum a_n$ and $\sum b_n$ are series with ______________________ terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then __________________________________.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is finite and $c > 0$, then __________________________________.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then __________________________________.

The Alternating Series Test:

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$, with $b_n > 0$ satisfies:

(i) ______________________ and (ii) ______________________

then the series is convergent.

The Ratio Test:

(i) If __________________________________, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and therefore convergent.

11. Using series tests. [20pts]

Determine whether the series converges or diverges. Justify your answers using one or more of the 7 series tests. Clearly state the name of test that you use and your conclusion.

(a) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+5}\right)$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n\sqrt{n}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ Hint: Compare to the harmonic series.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{100^n}{n!}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+2}}{n^3+n+1}$

12. Power Series Representations. [8pts]

a) Find a power series representation for the function $f(x) = \dfrac{x^3}{2-3x}$.

b) Find the interval of convergence of the power series in part a).

13. Maclaurin Series. [12pts]

a)Find the Maclaurin series for the function $f(x) = e^{-x^4}$.

(b) Use the power series you found above to evaluate $\int_0^1 e^{-x^4}\,dx$.

(c) Estimate $\int_0^1 e^{-x^4}\,dx$ using the Alternating Series Estimation Theorem so that the error less than $\frac{1}{400}$. You do not need to simplify the arithmetic in your approximation.

14. Taylor Series. [12pts]

a) Find the degree 3 Taylor polynomial, $T_3(x)$, for $f(x) = \sqrt[3]{x}$ centered at $a = 27$.

b) Use $T_3(x)$ to approximate $\sqrt[3]{26}$ and give the best bound possible on the error according to Taylors approximation theorem. Note, the Taylors Approximation Theorem is written on the next page.

Taylors Approximation Theorem:

If $|f^{(k+1)}(x)| \leq M$ for $|x-a| \leq d$,

then $|R_k(x)| \leq \dfrac{M}{(k+1)!}|x-a|^{k+1}$ for $|x-a| \leq d$.