The Signed Weighted Resolution Set is Not a Complete Pseudoknot Invariant

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Abstract

When the signed weighted resolution set (or were-set) was defined as an invariant of pseudoknots, it was unknown whether this invariant was complete. Using the Gauss-diagrammatic invariants of pseudoknots introduced by Dorais et al, we show that the signed were-set cannot distinguish all non-equivalent pseudoknots. This goal is achieved through studying the effects of a flype-like local move on a pseudodiagram.

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1 Introduction

1.1 Pseudodiagrams, Pseudoknots and Gauss Diagrams

Originally introduced by Hanaki in [2], pseudodiagrams are knot or link diagrams where some of the crossing information is missing. Where there is missing information, instead of a crossing with clearly marked over- and under-strands, a *precrossing* or double-point of the curve appears in the diagram.

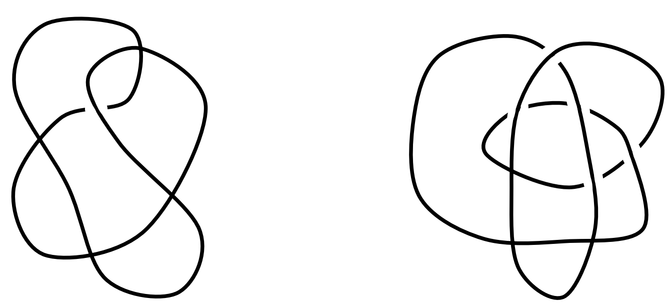

Figure 1: **Examples of pseudodiagrams**

How should we view this lack of crossing information? While flat crossings (as we find in the theories of flat virtual knots [6] or virtual strings [8]) are often pictured in the same way, precrossings ought to be interpreted differently. At a precrossing, we might say we are unsure about whether a given strand goes over or under at the crossing. In effect, precrossings behave somewhat like rigid

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vertices (as in [5]) or singular crossings (as in [9]). The primary difference is that in the theories of rigid vertex graphs in three-space and singular knots, no Reidemeister 1 type move is allowed, while an R1 type move should be allowed on precrossings. In contrast, a flat crossing can be interpreted as a positive or negative crossing in a knot theory where positive and negative crossings are taken to be equivalent. As a result, flat crossings are more manipulatable than precrossings. Thus, rigid vertices or singular crossings provide a much better reference for how a precrossing should be viewed.

Using this interpretation of precrossings, pseudoknots were first defined in [3] as equivalence classes of pseudodiagrams up to planar isotopy and a collection of natural Reidemeister moves. This collection of moves includes the classical Reidemeister (R) moves and a number of additional pseudo-Reidemeister (PR) moves as seen in Figure 2. The pseudo-Reidemeister 2 and 3 moves that are allowed are essentially those moves that would be allowed in classical knot theory if the precrossing appearing before the move and the precrossing appearing after the move were both replaced by positive crossings or were both replaced by negative crossings. The PR1 move is included because, regardless of whether the precrossing is replaced by a positive crossing or a negative crossing, a classical R1 move is allowable.

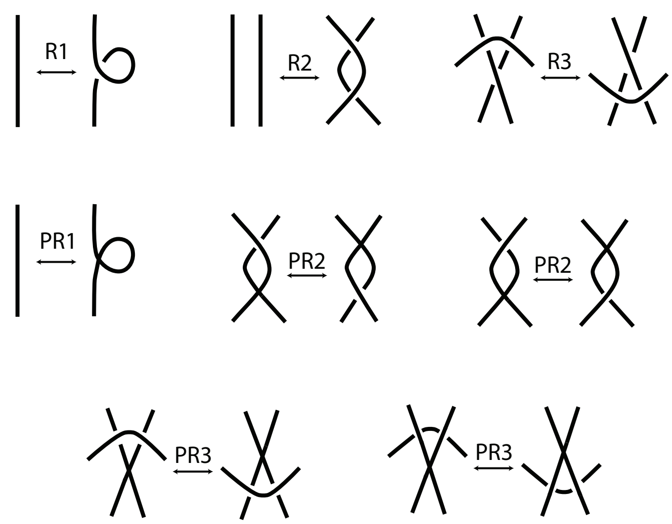

Figure 2: **Classical and Pseudo-Reidemeister moves**

While the most familiar representation of a given oriented classical knot is that of a knot diagram, an alternate and sometimes more useful representation is a Gauss diagram. A Gauss diagram consists of a core circle oriented counterclockwise (drawn to represent the entire curve of the oriented knot) together with a set of chords which connect the pre-image of double points from the knot diagram in the corresponding Gauss diagram. The crossing information is indicated on the chord by an arrow pointing from the over-strand to the under-strand and a sign on the chord specifying whether the crossing is positive or negative. See Figure 3.

In [1], the definition of a Gauss diagram was extended to pseudodiagrams as follows. All classical crossings in a pseudodiagram are represented in the Gauss diagram by a standard chord decorated with an arrow and a crossing sign. A precrossing is represented by a bold or thicker chord. We must take care, however, to indicate the proper 'handedness' of the precrossing. Thus, conventions similar to those introduced in [8] are adopted for precrossing chords. Figure 4 indicates a general precrossing and its decorated arrow within the Gauss diagram, and Figure 5 gives an example of a pseudoknot and its Gauss diagram. Notice that in the Gauss diagram, the precrossing arrow points in the same direction as the classical arrow would point if the precrossing were resolved positively

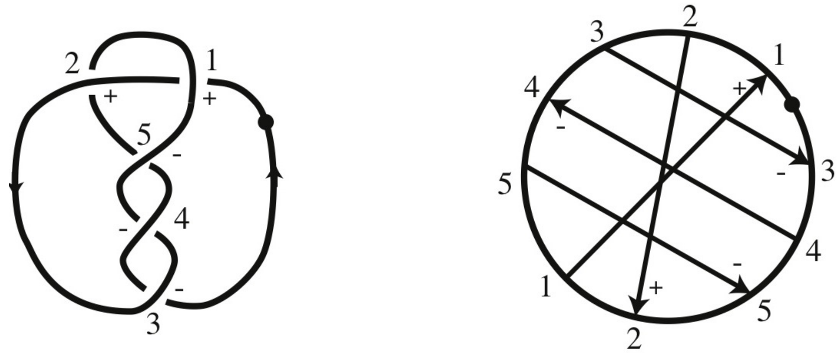

Figure 3: **A knot diagram and its corresponding Gauss diagram**

(i.e. replaced by a positive crossing).

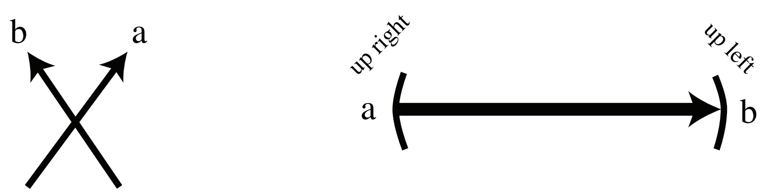

Figure 4: **A precrossing and a subsection of its Gauss diagram**

We refer the reader to [1] for illustrations of Gauss-diagrammatic versions of the R and PR moves. We note that not all Gauss diagrams correspond to pseudoknots, but the set of Gauss diagrams corresponds to a broader collection of knots referred to as *virtual pseudoknots*.

1.2 Invariants of Pseudoknots

There are two powerful pseudoknot invariants we focus our attention on in this paper, the signed were-set, introduced in [3], and the Gauss-diagrammatic invariant $\mathcal{I}(P)$, defined in [1]. We will use the latter invariant to show that the former invariant is incomplete.

Definition 1 *The* signed weighted resolution set *(or* were-set*) of a pseudoknot P is the set of ordered pairs (K, p_K) where K is the knot type of a resolution of P and p_K is the probability that K is obtained from P by a random choice of crossing information, assuming that positive and negative crossings are equally likely. In this set, knots and their mirror images are treated as distinct unless such a pair of knots are actually topologically equivalent.*

Definition 2 *Consider a Gauss diagram of a (virtual or classical) pseudoknot, G. Define a map $\mathcal{I}(G)$ as follows.*

1. *Replace with chords all arrows in G that are associated to precrossings. (I.e., delete all arrowheads on precrossing arrows.) These chords will be called* prechords.

2. *Decorate each prechord c with the integer value $i(c)$, where $i(c)$ is the sum of the signs of the classical arrows that intersect c.*

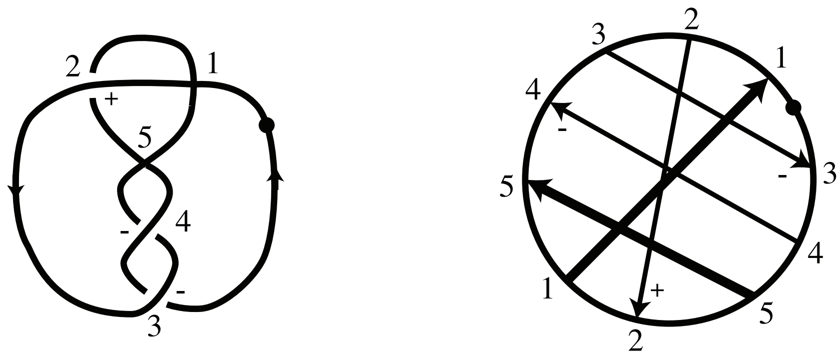

Figure 5: **A pseudoknot diagram and its corresponding Gauss diagram**

3. Delete all classical arrows.

4. Delete any prechords c that have adjacent endpoints and $i(c) = 0$.

The codomain of $\mathcal{I}$ is the set of all chord diagrams such that each chord is decorated with an integer. We refer to this set as $\mathbb{Z}\mathcal{C}$.

We illustrate this definition with an example of a virtual pseudoknot P and its corresponding decorated chord diagram $\mathcal{I}(P)$ in Figure 6.

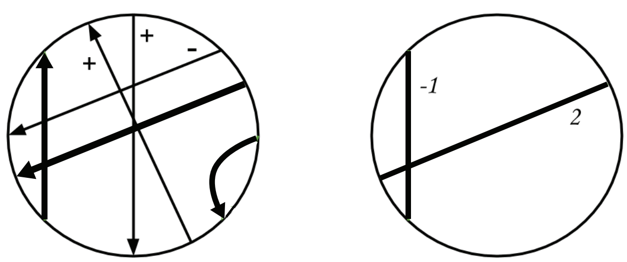

Figure 6: **A Gauss diagram for a virtual pseudoknot and its image under the map $\mathcal{I}$**

2 Shadow Flypes and the Were-Set

2.1 The Were-Set is Incomplete

To show that the were-set is incomplete, consider the pair of shadows that define two pseudoknots in Figure 7. We'll refer to these shadows as P_1 and P_2.

By construction, P_1 and P_2 are related by a move we will call a *shadow flype*, pictured in Figure 8. T is the shadow of a tangle in this figure. (This means that none of the crossings in the pseudotangle have been determined.)

In Figure 9, we see the values of $\mathcal{I}$ for P_1 and P_2. Because any equivalences between the values of $\mathcal{I}$ must preserve the cyclic ordering of the endpoints of chords in the chord diagrams, the values of $\mathcal{I}$ are distinct. Thus, P_1 and P_2. are distinct.

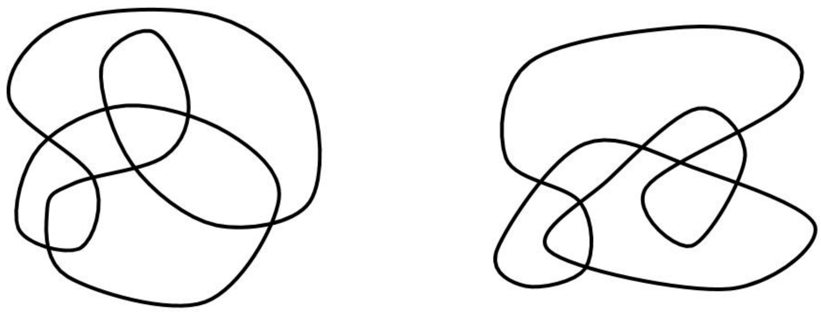

Figure 7: **A pair of nonequivalent pseudoknots, P_1 and P_2.**

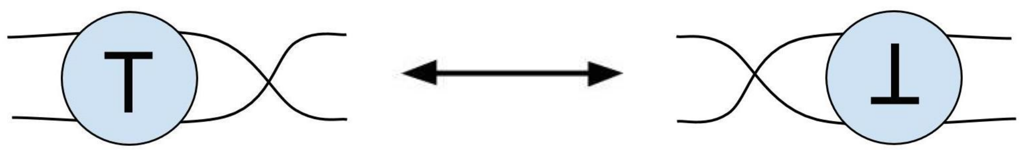

Figure 8: **A shadow flype.**

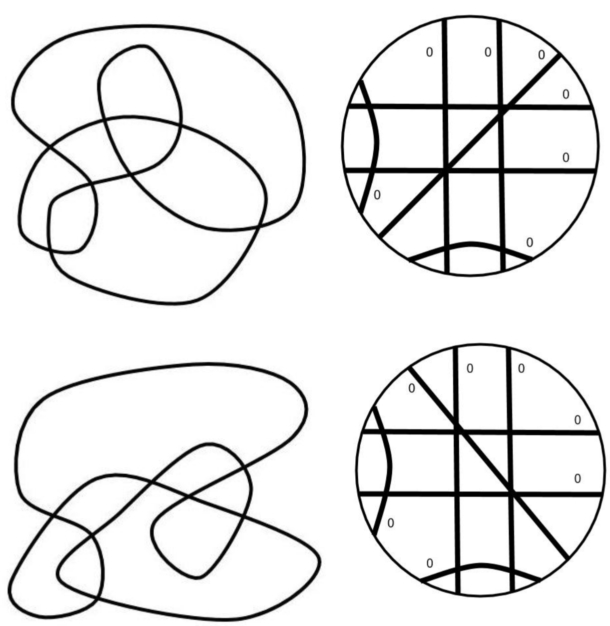

Figure 9: **Shadows P_1 and P_2 together with $\mathcal{I}(P_1)$ and $\mathcal{I}(P_2)$.**

What is interesting about this example is that for both pseudoknots, P_1 and P_2, the signed were-sets are the same. Computed by *LinKnot* [4], they are:

$$\{\{0_1, 72\}, \{-3_1, 10\}, \{3_1, 10\}, \{4_1, 20\}, \{-5_1, 1\}, \{-5_2, 2\},$$
$$\{5_1, 1\}, \{5_2, 2\}, \{6_2, 2\}, \{6_1, 2\}, \{-6_1, 2\}, \{-6_2, 2\}, \{-7_7, 1\}, \{7_7, 1\}\}.$$

(Note: the mirror image of knot K is denoted by $-K$.)

In fact, it is true in general that the were-set is unchanged by the shadow flype.

Theorem 1 *Suppose P_1 and P_2 are related by a local shadow flype move. Then the were-sets of P_1 and P_2 are identical.*

Proof. Suppose P_1 and P_2 are related by a local shadow flype move. Then there is a one-to-one correspondence between resolutions of P_1 and resolutions of P_2, where a knot diagram D_1 obtained by resolving precrossings of P_1 is paired with a resolution D_2 of P_2 such that D_1 and D_2 are related by a single (classical) flype. Thus, D_1 and D_2 represent equivalent knots. In effect, for any resolution of P_1, we can resolve the crossings of P_2 such that all crossings not involved in the shadow flype agree, and crossings involved in the shadow flype are chosen so that the resulting diagram is related to the resolution of P_1 by a classical flype. The existence of such a correspondence establishes our desired result. □

Theorem 1, together with our example provide the proof of our main result:

Theorem 2 *The signed were-set is not a complete invariant of pseudoknots.*

2.2 The Effect of a Shadow Flype & Examples

We just saw for a specific example how the shadow flype affects a pseudodiagram, and we used the Gauss-diagrammatic $\mathcal{I}$ invariant to illustrate that this transformation changed the pseudoknot type. We'd like to explore these effects more generally. How does a shadow flype transform the Gauss diagram of a pseudodiagram? Secondly, how can the pair of examples in Figure 7 be generalized to an infinite family of examples?

In [7], Soulié determines the effect on a Gauss diagram of performing a flype move on an ordinary knot. We can immediately derive from his work the effect of a shadow flype on the Gauss diagram G of a pseudoknot P. There are two possibilities for how the Gauss diagram of a pseudoknot might be affected, illustrated in Figure 10. These diagrams should be interpreted as follows: in each horizontal (vertical) shaded region, there may be chords whose endpoints both lie within that region. Arrows are omitted here since the direction of precrossing arrows does not affect the value of $\mathcal{I}(G)$. Thus, we will limit our focus to considering shadow flypes on the level of chord diagrams.

To understand how the diagrams in Figure 10 apply to an example, we provide the chord diagrams for shadows P_1 and P_2 in Figure 11. As we can see, this pair of examples illustrates a Type II flype. It is not difficult to prove that this pair of examples is the smallest counterexample to the signed were-set conjecture that can be obtained from a shadow flype. This follows from two facts: (1) in order for a chord diagram to be obtainable from a classical pseudoknot, each chord must intersect an even number of chords, and (2) any two chord diagrams that are related by a planar isotopy are equivalent.

Finally, let us observe that the P_1, P_2 counterexample can be generalized to an infinite family of counterexamples, illustrated in Figure 12. Note that both m and n must be even for these pseudoknots to be non-virtual. The values of $\mathcal{I}$ corresponding to these pseudoknots can be obtained from the chord diagrams shown simply by decorating each chord with a 0.

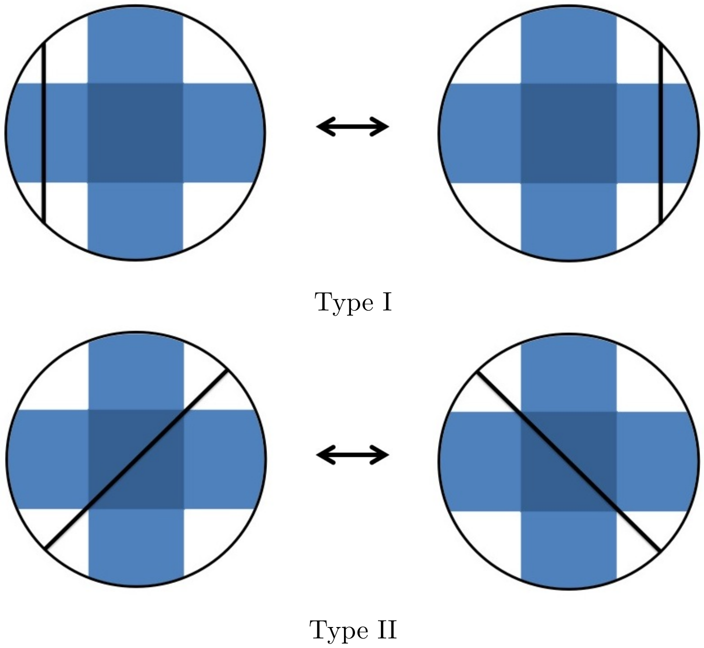

Figure 10: **Two types of flype effects on Gauss diagrams.**

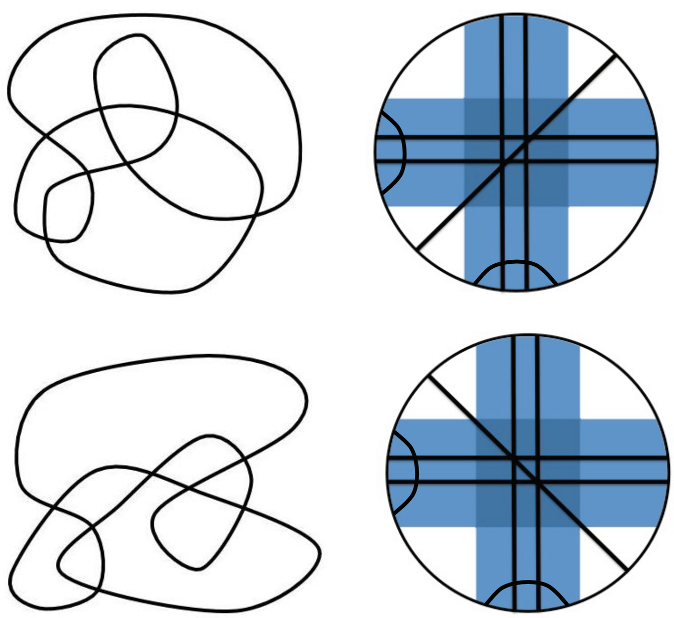

Figure 11: **Shadows P_1 and P_2 together with their Gauss diagrams.**

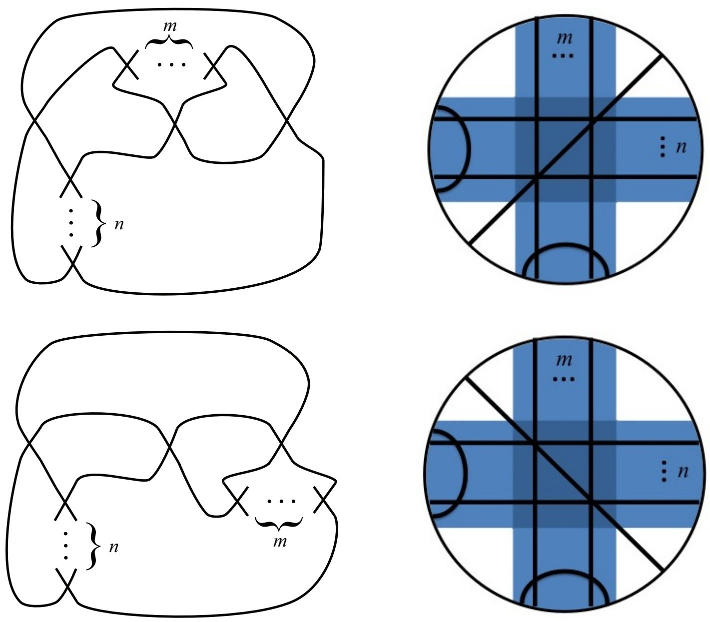

Figure 12: **An infinite family of pairs of counterexamples.**

2.3 Related Combinatorial Questions

The one open question related to this work that we are most interested in considering in the future is how many *non-equivalent* shadows can be obtained from a given shadow by shadow-flyping. Symmetries within chord diagrams may have the effect of producing equivalent pre- and post-flype diagrams, so not every possible flype move produces a new pseudoknot. This is an area for future exploration.

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