Group Examinations in Introduction-to-Proof Courses

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Abstract

We discuss the use of in-class group exams as a method of instruction, peer-review, and assessment. We describe our experience implementing these exams in an introduction-to-proof course designed primarily for math majors and minors. In our model, each of the group members has a unique proposition he or she is responsible for proving. Group members peer review each other's work, and an individual grade is assigned to each student based on the quality of the student's proof as well as that of his or her group members. This process gives students valuable experience actively creating and analyzing proofs, while introducing a second level of critical review and feedback. Because students are responsible for their own proofs as well as determining the validity of the work of their peers, these exams provide assessment of content knowledge, proof-writing skills, and reading comprehension of proofs. In this article, we discuss what makes a good group exam, and summarize student reaction to this form of assessment. We provide specific examples of exam questions and comments on how to write a group exam.

1 Background and Context

In mathematics, the act of writing a proof involves both *figuring out* a logical argument, and effectively *communicating* this argument to an audience. The goal of an introduction-to-proof course is to teach students both facets of the proof-writing process. But how do you assess student progress on these two fronts? In our experience, homework, though valuable, has several disadvantages as an assessment tool. When students work together, there is a likelihood of over-dependence on peers. On the other hand, while working alone, they do not receive immediate feedback, and misconceptions fester. At our institution, individual, in-class midterms are limited by time constraints, making it difficult for students to both figure out and craft proofs of more than two or three unfamiliar statements during the exam period. Thus, if we ask them to prove challenging statements, or statements which are unlike those they have proven before, we run the risk of students failing the entire exam because they became stuck on one problem. This high-stakes testing can lead to anxiety and loss of confidence. However, if we don't ask students to prove difficult or unfamiliar statements, how much of the *figuring out* process are we assessing?

The group exam format described herein uses peer discussion and solution-vetting to resolve these issues. Given the ability to brainstorm ideas, students are able to tackle challenging and novel proofs in the exam setting. Cooperative-learning activities such as this have been shown to increase student learning and retention of course content [5, 2]; and students self-report reduced test anxiety, elevated confidence, benefits from seeing the problem-solving techniques of others, deeper critical thinking, and increased enjoyment of the course [4, 3]. The group exam model we describe is unique in that it includes the benefits of collaboration, while assessing individual achievement and minimizing social loafing.

We have successfully implemented group exams at a large state school, a private R1 university, and our current school, Willamette University, a selective liberal arts college in Salem, Oregon. Willamette enrolls approximately 2000 undergraduate students and graduates approximately 12 math majors and 9 math minors per year. In this article we discuss our experiences using group exams in Willamette's introduction-to-proof course, Foundations of Advanced Mathematics. This course is nominally capped at 15 students, but enrollments have reached as high as 20 students in recent years. Foundations of Advanced Math is a prerequisite for all advanced math courses and is usually taken by math majors and minors in the spring of their freshman year after completing multivariable or integral calculus. It is in Foundations of Advanced Math where many of our students first encounter group exams, where they are given in addition to weekly homework and individual midterm and final exams.

2 Description and Implementation

A sample group exam with instructor notes is included in the Appendix. Below we describe the exam process, including preparation and grading, followed by a discussion of what makes a good group-exam problem.

On exam day, working in groups of three, students are given 60 minutes to complete a three-page exam. Each group member is responsible for proving a unique set of propositions from one page of the exam. Typically, students spend the first third to half of the exam period working on their individual exam page and quietly discussing with their group members any points which need clarification. For the remainder of the period, students actively share their work and review the work of their peers. They read critically to ensure the argument is valid and to give feedback on the clarity of their peers' writing. Students must then assess the suggestions made by their group members and incorporate this feedback into their own work accordingly. Students are only allowed to write on their individual exam pages, but are encouraged to collaborate with each other through discussion of all exam problems. Once students are satisfied that a group member's proof is valid and clearly written, they sign their names at the bottom of that group member's exam. At the end of the exam, all three group members should know how to prove the propositions on all three pages of the exam.

We incentivize and hold students accountable for their exam preparation by requiring that each student bring to the exam a written page of notes summarizing the current class material. A thorough page of notes benefits students directly as it may be referenced by all group members during the exam. The note page is turned in with the student's completed exam and counts for a small portion of his or her exam grade. Making this page of notes a requirement minimizes the likelihood of social loafing during the exam, or at the very least makes ill-prepared students easy to identify. Note pages can include statements of important theorems, definitions, and examples from class, but examples of completed proofs are not allowed. Students are informed that note pages and exam scores will be used to make future group assignments. In particular, we emphasize that students who are less prepared will be grouped with other less prepared students on subsequent exams. Knowing their future group assignments are based on their exam preparedness provides additional incentive for students to study for the group exam.

The grading process is two-fold, but not double the work. Each group exam is graded first on its individual merit, then it is assessed for peer-review points. We assign roughly 75% of the exam score based on the student's individual work and the remaining 25% on his or her peer-review work, which emphasizes to students that, although it is a *group* exam, they will be held accountable for their individual work. This distribution of points addresses a common student discomfort with group grading where all students earn the same grade though the individual quality of student work may vary. A student's peer-review points are lost when a group member makes a mistake that careful proofreading should have caught. We separate errors on the exam into two types: gross errors, such as conceptual or structural flaws, and subtle detail errors, for example minor calculation or small grammatical errors. Students lose up to half the points lost by their group member for each mistake of the former type and few points, if any, are lost for the latter. More specifically, missing a logical step or skipping a key justification that reduces a student's individual score by two points results in his group members each losing one peer-review point. Minor mistakes in algebra, calculation, or grammar are often at a level more detailed than that which the peer-reviewer has time to catch, so they rarely lose points for missing such errors. Group members do not lose points if a member of the group simply does not know how to do the proof and leaves it blank. Each team member is expected to pull his or her own weight. If a group member is stuck on his or her proof, the group is encouraged to work together to find a solution, but is not required to do so. If there is an unresolved disagreement about the validity of a proof, group members may sign partially down the exam page indicating the portion of the proof with which they agree. However, if the remainder of the proof is correct, students will lose peer-review points for not recognizing its validity. The collaborative nature of the exam and the instructor's guidance make disagreements and blank papers rare.

The instructor's role during the exam is active. We circulate through the class checking in with each group, especially during the first group exam when students are still gaining familiarity with the format. While we do not offer the kind of direct help we would provide on an in-class worksheet, we often facilitate group discussion and, when necessary, mediate disagreements by asking questions or suggesting questions that students could ask their group members. We frequently remind students of the time remaining and where they should be in the exam process. Proofreading and discussion take significantly longer than students expect. Consequently, time management is an important factor in successfully completing the exam.

When writing a group exam we carefully consider problem difficulty and length, in addition to content coverage. Since each group member has a different set of propositions to prove, it is important that the three pages of the exam be similar in length and difficulty while not overlapping significantly in content. We admit this is not an exact science. However, we try to assign three proofs that are challenging yet accessible with each having a degree of newness that makes it unlike any of the proofs students have seen before. One way to approach the group exam writing process is to reserve certain exercises, such as those that combine two topics in a new way, by not including them in lecture or homework so that they may be used on the exam. In this way, though each group exam takes 60 minutes of in-class time, we have found that they do not lead to a large loss in content coverage since the problems included would have been discussed in-class if not introduced on the exam. The sample group exam included in the Appendix covers function image, preimage, composition, and inverses. At the time of the exam, these topics had only recently been introduced to the students. In our course, we study functions after sets and power sets. These particular questions come from our course textbook, *Proofs and Fundamentals* by Ethan D. Bloch [1], which contains a wealth of good exercises. These particular problems had not been assigned as homework or covered in class.

3 Outcomes

The strengths of the group-exam model stem from the incorporation of reading and vetting into the exam setting. Our students' writing is improved by the authentic and immediate feedback they receive from peers. Questions we commonly overhear, such as "What do you mean here?" or "Why can you conclude that?", alert proof-writers to a lack of justification or a logical incoherence that they are often unable to recognize in their own writing. In our experience, comments from peers increase the proof-writer's attention to detail and explanation. Students quickly and naturally realize the importance of clarity in their writing. The vetting process makes students better able to appreciate the communicative power of a formal proof and less likely to view the proof-writing process as a series of hoops.

Another key outcome of group exams is students' increased ability to critically review proofs. In analyzing the proofs of their peers, students are exposed to authentic examples of potentially invalid proofs as well as a variety of writing styles and approaches. Not having previously considered the propositions proved by their group members, students must learn from their peers' writing and follow the argument presented. By necessity students must approach these proofs with greater skepticism, employing both critical-reading and analysis skills. On our individual exams (midterms and final) we include a problem asking students to read and assess the validity of a mathematical argument. Students often struggle with these problems. However, in courses with group exams, we have observed stronger performance by the final exam. In processing feedback from their group members, proof-writers engage in a second level of critical review. Because this feedback is not from an authority figure, students must determine the appropriateness and validity of the suggestions made. Furthermore, the close reading of their peers' proofs enhances students' ability to critically reflect on their own writing.

Gains in the quality of student proof-writing, reading, and analysis are reflected in higher individual cumulative final exam scores. Table 1 shows the average final exam scores for eight sections of Foundations of Advanced Mathematics taught by the authors. At least 75% of the final exam score is based on proof-writing problems (or problems asking students to prove or disprove a statement). Thus our final exams are a good indicator of a student's individual proof-writing ability. While we have an admittedly small sample of classes for comparison, there is a noticeable improvement in the exam averages of the five sections taught since implementing group exams.

Table 1: Section averages on Foundations of Advanced Math cumulative final exams

Sections without group exams			Sections with group exams				
68%	77%	63%	84%	79%	79%	73%	84%

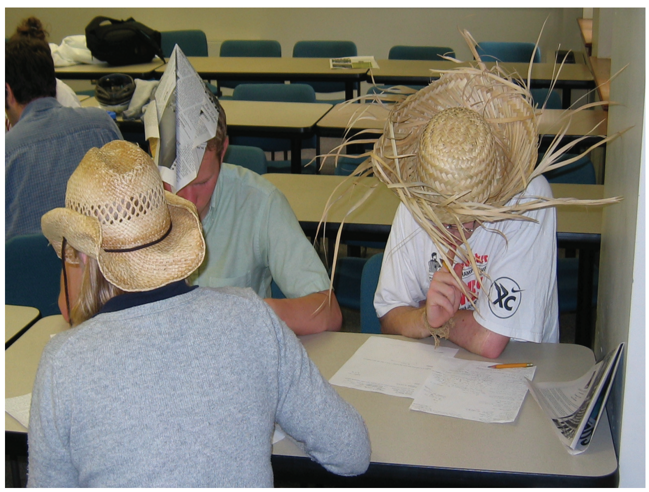

Figure 1: Students showing solidarity while working on a group exam.

The timed aspect of the exam forces nearly all conversation to be on-task. We have observed a substantial improvement in students' mathematical verbalization skills since implementing group exams. The specifics of what students say are predominately heard only by their group members; thus, there is less risk involved in contributing to the discussion than in speaking before the class as a whole. Having gained confidence explaining complex mathematical ideas in the small group setting, students are more likely to raise questions in class and take part in class discussions. The verbal practice students gain during the exam results in an improved ability to articulate questions, explain answers, and pinpoint areas of confusion. This improvement in students' communication skills leads to more meaningful class discussions, more productive conversations during office hours, and a noticeable improvement in the quality of their written proofs.

Since implementing group exams the class climate has become more cohesive, supportive, and active. From the instructor's perspective, group exams help us get to know our students better. Circulating through the class during the exam, we can give personalized encouragement and guidance while gaining a better understanding of our students' learning styles and misconceptions. Student responses to this exam format have been overwhelmingly positive. Students report increased confidence in the material and enjoyment of the course. The group-exam period is a time for both learning and assessment. Students expect and value the contributions of their peers. We have not observed the same level of collaboration and cooperation in classes without group exams, even when compared to classes with group projects or student presentations. Students in classes with group exams view their peers and the instructor as advocates for their success.

Student presentations are another common component of proof-based courses, and are frequently used by both authors. Presentations, like group exams, provide the authentic peer-review experience of judging the validity of a proof. We have found that student response to in-class presentations is more positive in courses where there are also group exams. We believe this is largely due to the cohesive and supportive classroom climate which group exams promote. For instructors concerned about the amount of class time required to have every student present, group exams provide an effective alternative. In the more intimate, collaborative and less stage-like environment of the group exam, all students are given the opportunity to critically read, analyze and critique the proofs of their peers, and receive feedback on their own work.

4 Extending the Method

Group exams have been successfully used by the authors in classes ranging from calculus to upper-level math courses. Although proof-writing is not a component of most calculus courses, students in all math classes benefit from focused discussion and proofreading solutions. Even in introductory-level courses, group exams promote better written communication of mathematics. In courses beyond the introduction-to-proofs level, group exams give students the opportunity to grapple with complex mathematical content and engage in conversations resembling those of experienced mathematicians.

Our recent experience using group exams has been at Willamette University where small class sizes are the norm. However, prior to our arrival at Willamette, we have used them successfully at the University of Oregon, with class sizes of 35-40 students, and at the University of Rochester with large lectures where group exams were administered by teaching assistants in recitations with approximately 20 students. In larger classes, it may be difficult for instructors to check in with all of their students during the group exam. This may increase the likelihood of frustration felt by some students in groups that struggle to communicate effectively. Thus, in this setting, group exams may result in fewer of the class-climate benefits, but students will still benefit from discussion and peer review.

References

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[3] Kari Morgan, Jeanne Rothaupt, Bruce A. Cameron, and Karen C. Williams, “Group Exams in the Higher Education Classroom: Strategies and Support for Successful Implementation,” *NACTA Journal* 51 (2007), no. 4, 38–45.

[4] Philip G. Zimbardo, Lisa D. Butler, and Valerie A. Wolfe, “Cooperative College Examinations: More Gain, Less Pain When Students Share Information and Grades,” *The Journal of Experimental Education* 71 (2003), no. 2, 101–125.

[5] John F. Zipp, “Learning by Exams: The Impact of Two-Stage Cooperative Tests,” *Teaching Sociology* 35 (2007), no. 1, 62–76.

Appendix

A Sample Group Exam with Instructor Notes

Page 1.

Problem 1A. Let A be a non-empty set and $\mathcal{P}(A)$ be its power set. Consider the function $F : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by

$$F(X) = A - X$$

for all $X \in \mathcal{P}(A)$. Prove that F is its own inverse function.

Problem 1B. Find two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that neither f nor g is constant, but $f \circ g$ is a constant function.

Instructor's Note: While prior to the exam students have seen examples of inverse function proofs, they have not seen a problem combining inverse functions and power sets. Students must realize that the solution includes a set equality proof. Problem 1B was added to balance the length of the three pages.

Page 2.

Problem 2A. Let A and B be sets, let $C \subseteq A$, $S \subseteq B$ and let $f : A \to B$ be a function. Prove that $f(C) \subseteq S$ if and only if $C \subseteq f^{-1}(S)$.

Problem 2B. In this part of the problem we show that it is not possible to strengthen the result above.
(i) Find an example of a function $f : A \to B$ together with sets $C \subseteq A$ and $S \subseteq B$ such that $f(C) = S$ and $C \neq f^{-1}(S)$.
(ii) Find an example of a function $f : A \to B$ together with sets $C \subseteq A$ and $S \subseteq B$ such that $f^{-1}(S) = C$ and $S \neq f(C)$

Instructor's Note: While prior to the exam students have seen proofs involving images and subsets, and proofs involving preimages and subsets, they have not seen this relationship linking subsets and images to subsets and preimages. Part B highlights a common misconception.

Page 3.

Problem 3. Let $f : A \to B$ be a function. For each $b \in B$, define the set $Q_b \subseteq A$ as follows

$$Q_b = \{x \in A | x \in f^{-1}(\{b\})\}.$$

Prove the following results about the family of sets $\{Q_b\}_{b \in B}$
(i) Prove that $A = \bigcup_{b \in B} Q_b$.
(ii) Prove that if $Q_{b_1} \cap Q_{b_2} \neq \emptyset$ for some $b_1, b_2 \in B$, then $b_1 = b_2$.

Instructor's Note: While prior to the exam students have studied families of sets and inverse image, they have never considered a family of sets defined in terms of inverse image. This problem also foreshadows the concept of partitions.