1 Gauss Diagrams

A knot diagram is a good *visual* representation of a knot, but it is often useful to have a representation that is more *combinatorial* in nature such as a Gauss diagram of a knot. The process to create a Gauss diagram begins with determining the Gauss code of the knot.

The **Gauss code** of a knot diagram with crossings labeled 1 through n is a cyclically ordered, double occurrence list of the integers $1, \ldots, n$ indicating the order of the crossings encountered as the knot is traversed from a selected starting point along the knot. Each integer in the code is preceded with an O or U to indicate whether the curve goes over or under, respectively. For example, the Gauss code for the trefoil in Figure 1 is O1U2O3U1O2U3.

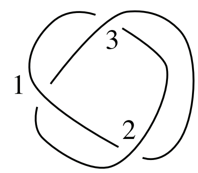

Figure 1: The Gauss code is O1U2O3U1O2U3

When creating the Gauss diagram, we simplify our view of the order of the crossings by wrapping the Gauss code around a circle, and we abstract the representation of the crossing by placing a chord of the circle between the two points along the circle that form the crossing. The U and O's in the Gauss code are dropped and replaced by arrows on the chords that point towards the under-crossing. Last, each oriented chord is decorated with + or - to indicate whether the crossing is positive or negative. An example of the Gauss diagram for the knot 6_3 is given in Figure 2.

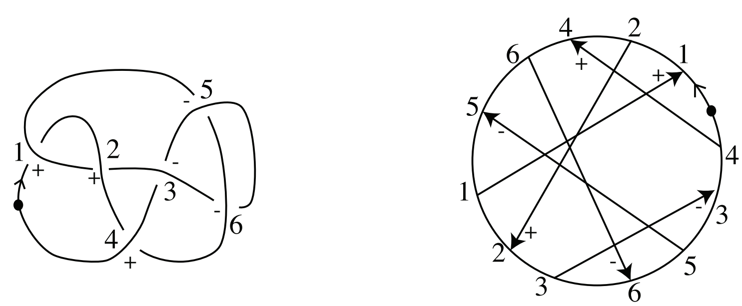

Figure 2: On the left 6_3 with Gauss code U1O2U4O6U5O1U2O3U6O5U3O4 and the corresponding Gauss diagram of 6_3 on the right.

The representation of crossings as chords in a Gauss diagram makes knot equivalence via Riedemeiseter moves less intuitive to detect visually. To aide our understand of knot equivalence in the setting of Gauss diagrams we translate the traditional Reidemeister moves to their Gauss diagrammatic equivalences. Figure 3 shows a generating set of Reidemeister 1, 2, and 3 moves. [1] Within the equivalences, the dotted portion of the circle could contain several other chords, but the solid portions of the circle contain nothing else.

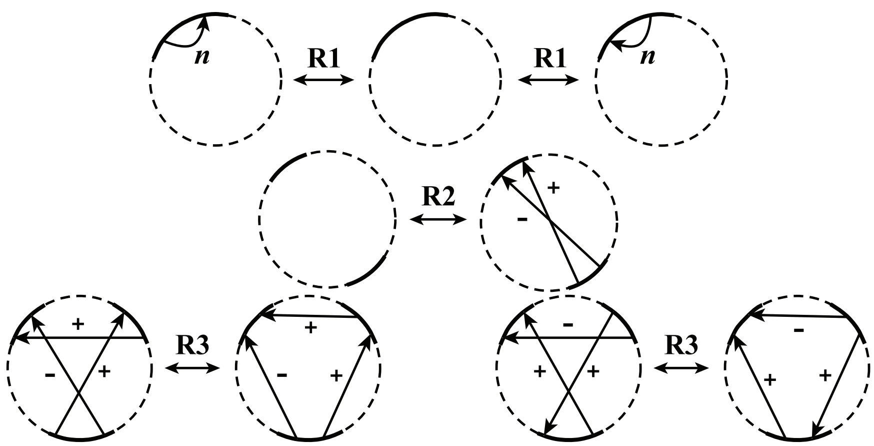

Figure 3: Reidemeister moves as Gauss diagrams. The n is a variable that may represent $+$ or $-$.

One observation that is straight forward to see is that a Gauss diagram without intersecting chords must represent the unknot. This fact follows from noticing that repeated application of the R1 Gauss move on the chords in the Gauss diagram that are exterior to other chords leaves us with a circle that contains no chords.

Another interesting feature of Gauss diagrams is that it is easy to draw a diagram for which there is no corresponding classical knot, see Figure 4. The theory of this broader collection of Gauss diagrams, for which there is no classical knot diagram, is called the theory of virtual knots and was pioneered independently by L. Kauffman [2] and N. & S. Kamada [3]. We learn more about virtual knots in Chapter *PutReferenceToChapter7Here* 7.

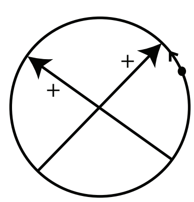

Figure 4: An example diagram that cannot be realized as the Gauss Diagram of a classical knot.

Gauss diagrams are a rich lens through which knots and related objects are studied. They are used to encode knots in a computer recognizable form. In Chapter *PutReferenceToChapter9Here*, we see how Gauss codes are used to study spatial graphs, and in Chapter *PutReferenceToChapter7Here*, we extend the definition of Gauss diagrams to pseudoknots use them to define invariants. When studying finite type invariants in Chapter *PutReferenceToChapter13Here*, we learn that any finite type invariant of classical knots is given by a Gauss diagram formula [4].

References

[1] M. Polyak, *Minimal generating sets of Reidemeister moves*, Quantum Topol., **1**, (2010), 399–411.

[2] L. H. Kauffman, *Virtual Knot Theory*, Europ. J. Combinatorics, 20, (1999), 663?691.

[3] N. Kamada and S. Kamada, *Abstract link diagrams and virtual knots*, J. Knot Theory Ramifications, 9: 1, (2000), 93–106.

[4] M. Goussarov, M. Polyak, O. Viro, *Finite-type invariants of classical and virtual knots*, Topology 39 (2000) 1045–1068