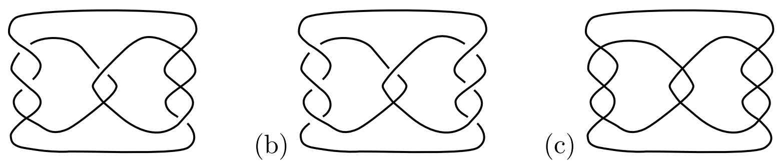
1 Pseudoknots and Singular Knots

In [1], Hanaki introduced the concept of a *pseudo diagram* as a generalization of a projection and a knot diagram. Specifically, a pseudo diagram is a diagram that contains crossings, where over/under information is provided, and pre-crossings, where over/under information is not specified. Examples of pseudo diagrams are seen in Figure 1.



(a) (b) (c)

Figure 1: Pseudo diagram examples

Before discussing pseudoknots, we make a few observations about pseudo diagrams. The *trivializing number* of a pseudo diagram P, denoted $\mathrm{tr}(P)$, is the least number of precrossings that must be resolved to guarantee the diagram is the unknot, regardless of how the remaining precrossings are resolved. For example, in Figure 1 (a), a careful consideration of cases shows that one pre-crossing in each twists of the pretzel diagram must be resolved to ensure the result is unknotted. Thus the trivializing number of diagram (a) is 3. If, as in (b) of Figure 1, there is no resolution of the pseudo diagram that results in the unknot, then the trivializing number is infinite. The *knotting number*, denoted $\mathrm{kn}(P)$ of a pseudo diagram is the minimum number of precrossings that must be resolved to guarantee the resulting diagram is knotted regardless of how the remaining precrossings are resolved. In (c) of Figure 1, resolving the three pre-crossing in the leftmost twist to be alternating results in at least a trefoil. Thus the knotting number of (c) is 3.

Pseudoknots are equivalence classes of pseudo diagrams where equivalence is defined by a sequence of ambient isotopies, the three classical Reidemeister moves, and the Pseudo Reidemeister moves shown in Figure 2. [2]

Singular knots, similar to pseudoknots, are knots that contain a finite number of self-intersections. We depict singular knots identically to pseudoknots with the pre-crossings viewed as singularities. The difference between the two is found in the definition of equivalence. Singular knot equivalence does not include the PR1 equivalence. For singular knots the number of singularities is an invariant, while for pseudoknots it is not. The reasoning behind the inclusion of the PR1 move in the theory of pseudoknots is that pre-crossings are viewed as crossings to be resolved, and in the case of the PR1 move, regardless of which crossing choice is selected, the knot type will remain the same. The origin of pseudoknots stems from the field of molecular biology where limited resolution pictures of knotted DNA results in images where some crossing information cannot be determined.

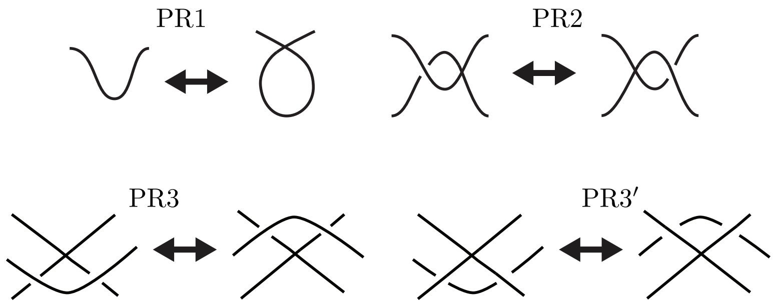


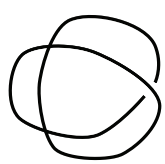
Figure 2: Pseudo Reidemeister moves

1.1 Pseudoknot Invariants

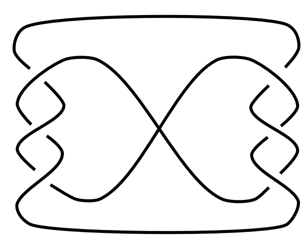
A natural question to ask when studying pseudoknots is what set of classical knots is obtained when the pre-crossings of a pseudoknot are resolved in all possible ways? The weighted resolution set, first studied in [2], is an invariant of pseudoknots that measures the answer to this question.

For a pseudoknot P, the *signed weighted resolution set* (or Were-set) of P is the set of ordered pairs (K, p_K) where K is a knot obtained by a resolution of all the pre-crossing of P and p_K is the probability that K is obtained from P via a random resolution of the crossings, assuming that positive and negative crossings are equally likely. For example, the signed weighted resolution set of the pseudoknot in Figure 3(a) is $\{(0_1, \frac{3}{4}), (-3_1, \frac{1}{4})\}$ because of the four possible resolutions of the two pre-crossings, three give the unknot and one results in the left-handed trefoil. Another example is shown in Figure 3(b). For this psuedoknot, the precrossing is resolved one way gives 6_1 and the other way gives -6_1.

Figure 3: Two pseudoknots and their Were-sets.



(a) Were-set $\{(0_1, \frac{3}{4}), (-3_1, \frac{1}{4})\}$



(b) Were-set $\{(6_1, \frac{1}{2}), (-6_1, \frac{1}{2})\}$.

The Were-Set of a pseudoknot is a powerful invariant and has been used to calculate a lower bound on the number of distinct equivalence classes of prime pseudoknots containing at least one pre-crossing but fewer than a total of 10 crossings plus pre-crossings, as shown in Table 1. For example, there are 10 distinct pseudoknots derived from the knot 5_2. [2]

Table 1: Number of distinct prime pseudoknots with crossing plus pre-crossing number n as measured by the Were-Set.[2]

n	3	4	5	6	7	8	9
number of pseudoknots	3	5	15	59	212	1344	7281

Unfortunately, the Were-Set is not a complete invariant of pseudoknots. Figure 4 contains two pseudoknots that differ by a local shadow flype move at the precrossing decorated by the grey disc. This flype relationship can be shown to give a one-to-one correspondence between equivalent knots in the resolution sets of these two pseudoknots. [3] Thus, the Were-Set invariant cannot distinguish between these two pseudoknots. However, using an pseudoknot invariant called the $\mathcal{I}$ invariant, we will see that the pseudoknots in Figure 4 are indeed distinct.

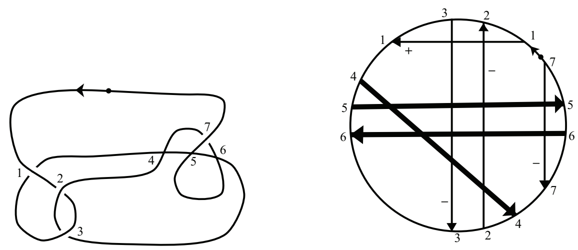
(a) P_1 (b) P_2

Figure 4: Both P_1 and P_2 have Were-set
$\{(0_1, \frac{72}{2^7}), (-3_1, \frac{10}{2^7}), (3_1, \frac{10}{2^7}), (4_1, \frac{20}{2^7}), (-5_1, \frac{1}{2^7}), (-5_2, \frac{2}{2^7}), (5_1, \frac{1}{2^7}), (5_2, \frac{2}{2^7}),$
$(6_2, \frac{2}{2^7}), (6_1, \frac{2}{2^7}), (-6_1, \frac{2}{2^7}), (-6_2, \frac{2}{2^7}), (-7_7, \frac{1}{2^7}), (7_7, \frac{1}{2^7})\}$

The invariant $\mathcal{I}$, first defined in [4], is an invariant applied to the Gauss diagram of pseudoknot. Recall, for a classical knot K, a knot diagram of K is an immersion of a circle in the plane, $f_K : S^1 \to \mathbb{R}^2$, such that each double point is decorated with crossing information. The Gauss diagram of a classical knot starts with the domain circle along with chords that connect the double points of f_K. Each chord is decorated with an arrow pointing toward the undercrossing and a sign designating the sign of the crossing. To extend Gauss diagrams to pseudoknots, the pre-crossings are represented by bolded or thicker chords. An arrow is given to each bolded arc that points to the undercrossing, if the pre-crossing were to be resolved positively. An example pseudoknot and its Gauss diagram are given in Figure 5.

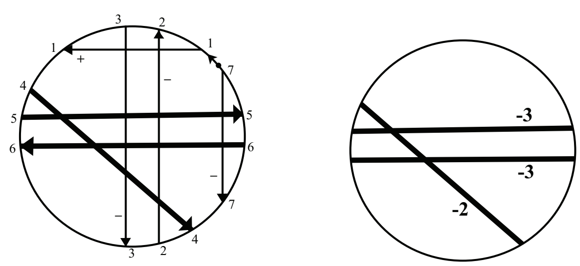
The invariant $\mathcal{I}$ is applied to the Gauss diagrams of a pseudoknot as follows: (1) remove the arrows on the bolded chords, (2) decorate each bolded chord with the integer value of the sum of the signs of the classical chords that it intersects

Figure 5: A pseudoknot and its Gauss diagram



in the chord diagram, (3) delete all classical chords and their decorations, (4) delete any bolded chords whose endpoints are adjacent along the circle and are decorated with a value of 0. The result is a circle with bolded chords decorated by integer values. The value of $\mathcal{I}(P)$ for the Gauss diagram from Figure 5 is shown in Figure 6.

Figure 6: A pseudoknot Gauss diagram, P, and the value of $\mathcal{I}(P)$



When the invariant $\mathcal{I}$ is applied to the two pseudoknots of Figure 4, we obtain the distinct values shown in Figure 7. This measured difference between the pseudoknots P_1 and P_2 is encoding the fact that the precrossing where the flype is performed (those with grey shading in Figure 4) can be resolved to a positive or a negative crossing. If the greyed precrossing in both P_1 and P_2 is resolved positively (denoted by P_1^+ and P_2^+), then the pseudoknot diagrams are equivalent by a flype move. Similarly, if they are both resolved negatively (denoted by P_1^- and P_2^-) the resulting pseudoknots are equivalent by a flype. But if they are resolved with opposite sign, then the $\mathcal{I}$ invariant detects that the resulting pseudoknots are distinct as shown in Figure 8.

1.2 Classical Invariants Extended to Pseudoknots

There are several classical invariants that have been studied in the context of pseudoknots such as p-colorability, knot determinant, and crossing number, to

Figure 7: The value of the $\mathcal{I}$ invariant for the two pseudoknots from Figure 4

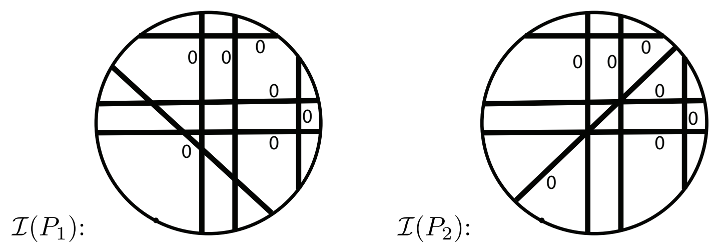


Figure 8: The $\mathcal{I}$ invariant applied to equivalent pseudoknots P_1^+ and P_2^+ and applied to P_1^- and P_2^-. The invariant shows P_1^+ is not equivalent to P_1^-.

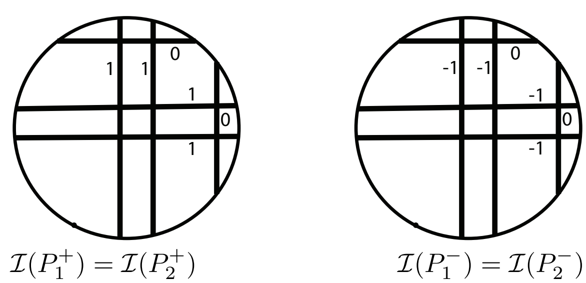
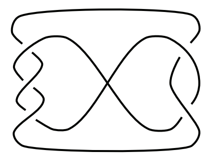


Figure 9: A pseudoknot with Were-set $\{(5_1, \frac{1}{2}), (5_2, \frac{1}{2})\}$, but crossing number 6.



name a few.

We begin with two seemingly distinct generalizations of p-colorability to pseudoknots first studied in [5] and extended in [6]. Similar to the classical setting, strands of a pseudoknot will begin and end only at a classical crossing. Therefore at a precrossing, all four pieces of the diagram emanating from the precrossing must be colored with the same value. A pseudoknot is *strong p-colorable* [5] if we can assign elements of $\mathbb{Z}/p\mathbb{Z}$ to the strands of the diagram so that at each classical crossing twice the number on the overstrand is equal mod-p to the sum of the values assigned to the understrands, and all arcs emanating from a precrossing of a pseudoknot must be assigned the same value. Alternatively, pseudoknot is said to be *p-colorable* if all of the resolutions of a pseudoknot are p-colorable. In [5] they show that strong p-colorable implies p-colorable, and in [6] they prove that for p an odd prime, p-colorable implies strong p-colorable. Thus for odd primes p, the two generalizations of p-colorability are in fact equivalent.

As is the case for classical knots, the concept of a knot determinant is used to determine for which p is a give pseudoknot p-colorable? The *pseudodeterminant* of a pseudoknot K is defined as the greatest common divisor of the determinants of all the resolutions of K and a pseudoknot is p-colorable for every value of p that divides its pseudodeterminant. The definition of p-colorability for pseudoknots is quite restrictive. In [5], the colorabiltiy of 8583 psuedoknots with 9 or fewer crossings is determined, only 112 are actually non-trivially colorable.

The *crossing number* of a pseudoknot K, denoted $cr(K)$, is defined as the minimum number of total crossings (both classical and precrossings) in any projection of K. At first glance, it might seem as though the crossing number of a pseudoknot ought to equal to the maximum crossing number of its resolutions. However, the example in Figure 9 shows this is not the case. However, if a pseudoknot contains no nugatory crossings and the precrossings could be resolved to result in an alternating diagram, then the crossing number of the pseudoknot is indeed the maximum crossing number of its resolutions. [2]

References

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